Schwinger Model with Quartic Interactions

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I. QUARTIC FERMIONIC INTERACTIONS

Since quartic fermionic interactions are marginal in the Schwinger model, nothing prohibits us to include them in the action. So, we can consider the terms:

$$
\begin{aligned}
\delta S_g &= g \int d^2x \ (\bar{\Psi}\Psi)^2 \\
\delta S_\lambda &= \lambda \int d^2x \ \bar{\Psi}\gamma^\mu\Psi \ \bar{\Psi}\gamma_\mu\Psi \\
\delta S_\beta &= \beta \int d^2x \ (\bar{\Psi}\gamma^3\Psi)^2 \\
\delta S_\kappa &= i\kappa \int d^2x \ A_\mu \bar{\Psi}\gamma^\mu\Psi \ \bar{\Psi}\Psi,
\end{aligned} \tag{1}
$$

using the Dirac representation of the Gamma matrices,

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^3 = \gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}
$$

And together with $\Psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$, we have that

$$
(\bar{\Psi}\Psi)^2 = \left[\begin{pmatrix} \psi_u^\dagger & \psi_d^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \right]^2 = (\psi_u^\dagger\psi_u)^2 + (\psi_d^\dagger\psi_d)^2 - (\psi_u^\dagger\psi_u)(\psi_d^\dagger\psi_d) - (\psi_d^\dagger\psi_d)(\psi_u^\dagger\psi_u) \tag{2}
$$

$$
(\bar{\Psi}\gamma^3\Psi)^2 = \left[\begin{pmatrix} \psi_u^\dagger & \psi_d^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \right]^2 = (\psi_u^\dagger\psi_d)^2 + (\psi_d^\dagger\psi_u)^2 - (\psi_u^\dagger\psi_d)(\psi_d^\dagger\psi_u) - (\psi_d^\dagger\psi_u)(\psi_u^\dagger\psi_d) \tag{3}
$$

$$
\begin{aligned}
(\bar{\Psi}\gamma^\mu\Psi)^2 &= \left[\begin{pmatrix} \psi_u^\dagger & \psi_d^\dagger \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \right]^2 - \left[\begin{pmatrix} \psi_u^\dagger & \psi_d^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \right]^2 = (\psi_u^\dagger\psi_u + \psi_d^\dagger\psi_d)^2 - (\psi_u^\dagger\psi_d + \psi_d^\dagger\psi_u)^2 \\
&= (\psi_u^\dagger\psi_u)^2 + (\psi_d^\dagger\psi_d)^2 - (\psi_u^\dagger\psi_d)^2 - (\psi_d^\dagger\psi_u)^2 + (\psi_u^\dagger\psi_u)(\psi_d^\dagger\psi_d) + (\psi_d^\dagger\psi_d)(\psi_u^\dagger\psi_u) - \\
&\quad - (\psi_u^\dagger\psi_d)(\psi_d^\dagger\psi_u) - (\psi_d^\dagger\psi_u)(\psi_u^\dagger\psi_d)
\end{aligned} \tag{4}
$$

And choosing the $A_0 = 0$ gauge fixing,

$$
\begin{aligned}
A_\mu \bar{\Psi}\gamma^\mu\Psi \ \bar{\Psi}\Psi &= A_1 \left[\begin{pmatrix} \psi_u^\dagger & \psi_d^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \right] \left[\begin{pmatrix} \psi_u^\dagger & \psi_d^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \right] \\
&= A_1 \left[(\psi_u^\dagger\psi_d)(\psi_u^\dagger\psi_u) - (\psi_u^\dagger\psi_d)(\psi_d^\dagger\psi_d) + (\psi_d^\dagger\psi_u)(\psi_u^\dagger\psi_u) - (\psi_d^\dagger\psi_u)(\psi_d^\dagger\psi_d) \right]
\end{aligned} \tag{5}
$$

If we define our previous operators in a normal ordered way (all daggered operators to the left), i.e. we avoid contact terms of the form $\delta(0)\psi^\dagger\psi$, then

$$
\begin{aligned}
&(\bar{\Psi}\Psi)^2 = -(\bar{\Psi}\gamma^3\Psi)^2 = -\frac{1}{2}(\bar{\Psi}\gamma^\mu\Psi)^2 = -\psi_u^\dagger\psi_d^\dagger\psi_u\psi_d - \psi_d^\dagger\psi_u^\dagger\psi_d\psi_u \\
&A_\mu \bar{\Psi}\gamma^\mu\Psi \ \bar{\Psi}\Psi = 0
\end{aligned} \tag{6}
$$

Note that an interaction of the form $(\bar{\Psi}\gamma^\mu\gamma^3\Psi)^2 = -(\bar{\Psi}\gamma^\mu\Psi)^2$ and the interaction $(\bar{\Psi}\gamma^3\Psi)(\bar{\Psi}\Psi)$ is proportional to the term $A_\mu(\bar{\Psi}\gamma^\mu\Psi)(\bar{\Psi}\Psi)$.

II. QUARTIC INTERACTION ON THE LATTICE

If we perform the staggered representation of the fermions, i.e. we put $\psi_u(x)$ on even sites and $\psi_d(x)$ on odd ones, we have the identification, where a is the lattice spacing:

$$\begin{aligned} \psi_u &= \frac{1}{\sqrt{a}}\, c_n, \quad \text{n-even} \\ \psi_d &= \frac{1}{\sqrt{a}}\, c_n, \quad \text{n-odd}, \end{aligned} \tag{7}$$

then the interacting Hamiltonians can be written as

$$\begin{aligned} H_g &= -\frac{g}{a} \sum_{n-\text{odd}} \left[(c^\dagger_{n+1} c_{n+1})^2 + (c^\dagger_n c_n)^2 - (c^\dagger_{n-1} c_{n-1})(c^\dagger_n c_n) - (c^\dagger_n c_n)(c^\dagger_{n+1} c_{n+1}) \right] \\ &= -\frac{g}{a} \sum_n \left[(c^\dagger_n c_n)^2 - (c^\dagger_n c_n)(c^\dagger_{n+1} c_{n+1}) \right] \end{aligned}$$

Now, performing the Jordan-Wigner transformation

$$c_n = \prod_{\ell<n} (i\sigma^z_\ell)\, \sigma^-_n\,, \qquad c^\dagger_n = \prod_{\ell<n} (-i\sigma^z_\ell)\, \sigma^+_n\,, \tag{8}$$

we get

$$\begin{aligned} H_g &= -\frac{g}{a} \sum_n \left[\frac{(1+\sigma^z_n)}{2} - \frac{(1+\sigma^z_n)(1+\sigma^z_{n+1})}{4} \right] \\ &= -\frac{g}{4a} \left[N - \sum_n \sigma^z_n \sigma^z_{n+1} \right] \end{aligned}$$

the first term is just an additive constant, so the interaction reads:

$$\begin{aligned} H_g &= \frac{g}{4a} \sum_n \sigma^z_n \sigma^z_{n+1}, \quad \text{or} \\ &\boxed{W_g = \frac{2}{e^2 a} H_g = \frac{g\ x}{2} \sum_n \sigma^z_n \sigma^z_{n+1}}. \end{aligned} \tag{9}$$

where $x = \frac{1}{e^2 a^2}$.

Similarly, we can obtain the H_λ term:

$$
\begin{aligned}
H_\lambda = & -\frac{\lambda}{a}\sum_{n-\text{odd}}\Big[(c^\dagger_{n+1}c_{n+1})^2 + (c^\dagger_n c_n)^2 - (c^\dagger_{n-1}c_n)^2 - (c^\dagger_n c_{n+1})^2 + (c^\dagger_{n-1}c_{n-1})(c^\dagger_n c_n) + \\
& \quad + (c^\dagger_n c_n)(c^\dagger_{n+1}c_{n+1}) - (c^\dagger_{n-1}c_n)(c^\dagger_n c_{n-1}) - (c^\dagger_n c_{n+1})(c^\dagger_{n+1}c_n)\Big] \\
= & -\frac{\lambda}{a}\sum_n \Big[(c^\dagger_n c_n)^2 - (c^\dagger_n c_{n+1})^2 + (c^\dagger_n c_n)(c^\dagger_{n+1}c_{n+1}) - (c^\dagger_n c_{n+1})(c^\dagger_{n+1}c_n)\Big] \\
= & -\frac{\lambda}{a}\sum_n \left[\frac{(1+\sigma^z_n)}{2} + (\sigma^+_n e^{i\theta_n}\sigma^-_{n+1})^2 + \frac{(1+\sigma^z_n)(1+\sigma^z_{n+1})}{4} - \sigma^+_n\sigma^-_n\sigma^+_{n+1}\sigma^-_{n+1}\right] \\
= & -\frac{\lambda}{a}\sum_n \left[\frac{(1+\sigma^z_n)}{2} + \frac{(1+\sigma^z_n)(1+\sigma^z_{n+1})}{4} - \frac{(1+\sigma^z_n)(1-\sigma^z_{n+1})}{4}\right] \\
= & -\frac{\lambda}{a}\sum_n \left[\frac{(1+\sigma^z_n)(1+\sigma^z_{n+1})}{2}\right] \\
= & -\frac{\lambda}{2a}\left[N + 4Q + (1-(-1)^N) + \sum_n \sigma^z_n\sigma^z_{n+1}\right], \qquad Q = \sum_n \frac{\sigma^z_n + (-1)^n}{2},
\end{aligned}
$$

where we have used that $(\sigma^+)^2 = (\sigma^-)^2 = 0$. Neglecting the constants, the interaction reads:

$$
\begin{gathered}
H_\lambda = -\frac{\lambda}{2a}\sum_n \sigma^z_n\sigma^z_{n+1}, \\
\boxed{W_\lambda = \frac{2}{e^2 a}H_\lambda = -\lambda\, x\sum_n \sigma^z_n\sigma^z_{n+1}}.
\end{gathered} \tag{10}
$$

We calculate the H_β term

$$
\begin{aligned}
H_\beta = & -\frac{\beta}{a}\sum_{n-\text{odd}}\Big[(c^\dagger_n c_{n+1})^2 + (c^\dagger_{n-1}c_n)^2 - (c^\dagger_{n-1}c_n)(c^\dagger_n c_{n-1}) - (c^\dagger_n c_{n+1})(c^\dagger_{n+1}c_n)\Big] \\
= & -\frac{\beta}{a}\sum_n \Big[(c^\dagger_n c_{n+1})^2 - (c^\dagger_n c_{n+1})(c^\dagger_{n+1}c_n)\Big] \\
= & -\frac{\beta}{a}\sum_n \big[(\sigma^+_n e^{i\theta_n}\sigma^-_{n+1})^2 - \sigma^+_n\sigma^-_{n+1}\sigma^+_{n+1}\sigma^-_n\big] \\
= & \frac{\beta}{a}\sum_n \left[\frac{(1+\sigma^z_n)(1-\sigma^z_{n+1})}{4}\right] \\
= & \frac{\beta}{4a}\left[N - \sum_n \sigma^z_n\sigma^z_{n+1}\right],
\end{aligned}
$$

ignoring the constant term, we have

$$
\begin{gathered}
H_\beta = -\frac{\beta}{4a}\sum_n \sigma^z_n\sigma^z_{n+1}, \quad \text{or} \\
\boxed{W_\beta = \frac{2}{e^2 a}H_\beta = -\frac{\beta\, x}{2}\sum_n \sigma^z_n\sigma^z_{n+1}}.
\end{gathered} \tag{11}
$$

For the H_κ contribution, we have

$$
\begin{aligned}
H_\kappa = & -i\frac{\kappa}{a}\sum_{n-\text{odd}}\Big[(c_n^\dagger e^{i\theta_n}c_{n+1})(c_{n+1}^\dagger c_{n+1})-(c_n^\dagger e^{i\theta_n}c_{n+1})(c_n^\dagger c_n)+\\
& +(c_{n+1}^\dagger e^{-i\theta_n}c_n)(c_{n+1}^\dagger c_{n+1})-(c_{n+1}^\dagger e^{-i\theta_n}c_n)(c_n^\dagger c_n)\Big]\\
= & -i\frac{\kappa}{2a}\sum_n\Big[(c_n^\dagger e^{i\theta_n}c_{n+1})(c_{n+1}^\dagger c_{n+1})-(c_n^\dagger e^{i\theta_n}c_{n+1})(c_n^\dagger c_n)+c.c.\Big]\\
= & -\frac{\kappa}{2a}\sum_n\left[\left(\sigma_n^+e^{i\theta_n}\sigma_{n+1}^-\right)\frac{(1+\sigma_{n+1}^z)}{2}-\left(\sigma_n^+e^{i\theta_n}\sigma_{n+1}^-\right)\frac{(1+\sigma_n^z)}{2}+c.c.\right]\\
= & -\frac{\kappa}{2a}\sum_n\left[\frac{1}{2}(\sigma_n^+e^{i\theta_n}\sigma_{n+1}^-\sigma_{n+1}^z)-\frac{1}{2}(\sigma_n^+\sigma_n^z e^{i\theta_n}\sigma_{n+1}^-)+c.c.\right]\\
= & -\frac{\kappa}{2a}\sum_n\left[\sigma_n^+e^{i\theta_n}\sigma_{n+1}^-+c.c.\right],
\end{aligned}
$$

where we have used that $\sigma^+\sigma^z=-\sigma^+$ and $\sigma^-\sigma^z=\sigma^-$. Therefore,

$$
\begin{aligned}
H_\kappa &= -\frac{\kappa}{2a}\sum_n\left[\sigma_n^+e^{i\theta_n}\sigma_{n+1}^-+c.c.\right],\\
&\boxed{W_\kappa=\frac{2}{e^2a}H_\kappa=-\kappa\, x\sum_n\left[\sigma_n^+e^{i\theta_n}\sigma_{n+1}^-+c.c.\right]}.
\end{aligned}
\tag{12}
$$

Since all the operators are related to each other (or are zero) when normal ordered, then the dimensionless Schwinger model Hamiltonian with quartic interactions is:

$$
W=\sum_n\left[L_n+\frac{\theta}{2\pi}\right]^2+\frac{\mu}{2}\sum_n(-1)^n\sigma_n^z+x\sum_n\left[\sigma_n^+e^{i\theta_n}\sigma_{n+1}^-+\text{c.c.}-\lambda\sigma_n^z\sigma_{n+1}^z\right],
\tag{13}
$$

where $\mu=\frac{2m_{latt}}{ae^2}$ and $m_{latt}=m-\frac{e^2a}{8}$.

III. MASSLESS CASE

If we consider the Schwinger model with zero fermion mass and quartic interactions, then the continuum action reads

$$
\begin{aligned}
S &= \int d^2x\left(\frac{1}{2}F_{01}^2+e\frac{\theta}{2\pi}F_{01}+i\bar\Psi\gamma^\mu D_\mu\Psi+\lambda\bar\Psi\gamma^\mu\Psi\ \bar\Psi\gamma_\mu\Psi\right)\\
&= \int d^2x\left(\frac{1}{2}F_{01}^2+e\frac{\theta}{2\pi}F_{01}+i\bar\Psi\gamma^\mu\partial_\mu\Psi+eA_\mu j_V^\mu+\lambda\bar\Psi\gamma^\mu\Psi\ \bar\Psi\gamma_\mu\Psi\right)\\
&= \int d^2x\left(\frac{1}{2}F_{01}^2+e\frac{\theta}{2\pi}F_{01}+\frac{1}{8\pi}(\partial_\mu\phi)^2+\frac{e}{2\pi}A_\mu\epsilon^{\mu\nu}\partial_\nu\phi+\frac{\lambda}{4\pi^2}\ (\partial^\mu\phi)^2\right)\\
&= \int d^2x\left(\frac{1}{2}F_{01}^2+e\frac{(\phi+\theta)}{2\pi}F_{01}+\frac{1}{8\pi}\left(1+\frac{2\lambda}{\pi}\right)(\partial_\mu\phi)^2\right)\\
&= \int d^2x\left(\frac{1}{8\pi}\left(1+\frac{2\lambda}{\pi}\right)(\partial_\mu\phi)^2-\frac{e^2}{8\pi^2}(\phi+\theta)^2\right)\\
&= \int d^2x\left(\frac{1}{2}(\partial_\mu\phi)^2-\frac{e^2}{2\pi}\left(1+\frac{2\lambda}{\pi}\right)^{-1}\phi^2\right),
\end{aligned}
\tag{14}
$$

where we can see that the Schwinger mass gets redefined by

$$M_S = \frac{e}{\sqrt{\pi + 2\lambda}} = \frac{e}{\sqrt{\pi - g}}. \tag{15}$$

Given that our effective theory is a free scalar one for every value of $\lambda > -\pi/2$, then λ is an exactly marginal operator (it doesn't run since there are no interactions in the low-energy action).

We can obtain a similar expression for small values of λ, using the asymptotic expansion for PBC, studied in [1]. But, one has to be aware that in this approach the limits are taken as $x \gg 1$ and then $N \to \infty$. This is not the correct order of taking the continuum extrapolation (the right procedure is first taking the thermodynamic limit and then sending $x \to \infty$). However, in this case it appears to work. So, if we take $x \gg 1$, then the leading term in the asymptotic expansion is obtained by neglecting the gauge fields. So, our Hamiltonian reduces to the one of the XXZ model[1]

$$\begin{aligned} W &\sim x \sum_n \left[\sigma_n^+ \sigma_{n+1}^- + \text{c.c.} - \lambda \sigma_n^z \sigma_{n+1}^z\right] \\ &= \frac{x}{2} \sum_n \left[X_n X_{n+1} + Y_n Y_{n+1} - 2\lambda Z_n Z_{n+1}\right] \\ &= \frac{x}{2} \sum_n \left[X_n X_{n+1} + Y_n Y_{n+1} + g Z_n Z_{n+1}\right], \end{aligned} \tag{16}$$

the ground energy of this model in the interval $-1 < g < 1$ is

$$\frac{\epsilon_0}{N} = 2x \left[-\frac{g}{4} + \frac{\sqrt{1-g^2}}{2} \int_{-\infty}^{\infty} dw \frac{\sinh\left((\pi - \gamma)w\right)}{\sinh\left(\pi w\right) \cosh\left(\gamma w\right)} \right], \tag{17}$$

where $g = \cos\gamma$. The next term in the asymptotic expansion is obtained by neglecting all the gauge fields (via Gauss law) but the average field:

$$\theta = \sum_{n=1}^{N} \frac{\theta(n)}{\sqrt{N}}.$$

Thus, the Hamiltonian can be read as:

$$\begin{aligned} W &\simeq -\frac{\partial^2}{\partial\theta^2} + \epsilon_0 \cos\frac{\theta}{\sqrt{N}} X \\ &\simeq -\frac{\partial^2}{\partial\theta^2} - \epsilon_0 \left(1 - \frac{1}{2}\left(\frac{\theta}{\sqrt{N}}\right)^2\right) X, \end{aligned}$$

with X being the Pauli matrix σ^x and in the second line we have expanded θ around one of the minima of the potential $\epsilon_0 \cos\frac{\theta}{\sqrt{N}}$, say $\theta = \sqrt{N}\pi$. This gives,

$$\begin{aligned} f_0 &\sim N e^{-\frac{\theta^2}{2\sqrt{N}}\sqrt{\epsilon_0/2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow E_0 = -\epsilon_0 + \sqrt{\frac{\epsilon_0}{2N}} \\ f_1 &\sim N\theta e^{-\frac{\theta^2}{2\sqrt{N}}\sqrt{\epsilon_0/2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow E_1 = -\epsilon_0 + 3\sqrt{\frac{\epsilon_0}{2N}}. \end{aligned}$$

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[1] Notice that, in comparison with [1], we have flipped the sign of x. In fact, one can check that the Hamiltonian $W(x, \lambda)$ has the same spectrum as $W(-x, -\lambda)$. This is justified by the action of the operator $P_1 = Z_1 Z_3 Z_5 \ldots$ (or similarly by the operator $P_2 = Z_2 Z_4 Z_6 \ldots$) as follows: $P_1 W(x, \lambda) P_1 = W(-x, -\lambda)$. Note that $P = P_1 P_2$ commutes with the Hamiltonian.

Hence,

$$\frac{E_1 - E_0}{2e\sqrt{x}} = \sqrt{-\frac{g}{4} + \frac{\sqrt{1-g^2}}{2}\int_{-\infty}^{\infty} dw \frac{\sinh\left((\pi-\gamma)w\right)}{\sinh(\pi w)\cosh(\gamma w)}}. \tag{18}$$

We can see that for $g = 0$, we recover the well-known Schwinger mass $1/\sqrt{\pi}$.

Now, we can compare by plotting (15) and (18) as shown in Fig. 1. Both functions are in very good agreement in the interval $-1 \ll g \ll 1$. If we use the same approach for the other intervals, the agreement is poor, as depicted in Fig. 2. We can show that if we expand around $g \simeq 0$, we get that $\gamma \simeq \frac{\pi}{2} - g$, and so (18) reduces to

$$\begin{aligned}
\frac{E_1 - E_0}{2e\sqrt{x}} &\simeq \sqrt{-\frac{g}{4} + \frac{1}{2}\int_{-\infty}^{\infty} dw \frac{\sinh\frac{\pi}{2}w + gw\cosh\frac{\pi}{2}w}{\sinh(\pi w)\left(\cosh\frac{\pi}{2}w - gw\sinh\frac{\pi}{2}w\right)}} \\
&\simeq \sqrt{-\frac{g}{4} + \frac{1}{2}\int_{-\infty}^{\infty} dw \left[\frac{1}{2\cosh^2\frac{\pi}{2}w} + gw\left(\frac{1}{\sinh\pi w} + \frac{\sinh\frac{\pi}{2}w}{2\cosh^3\frac{\pi}{2}w}\right)\right]} \\
&\simeq \sqrt{\frac{1}{\pi} + \frac{g}{\pi^2}} \\
&\simeq \frac{1}{\sqrt{\pi}}\left(1 + \frac{g}{2\pi}\right) \simeq \frac{M_S}{e}.
\end{aligned}$$

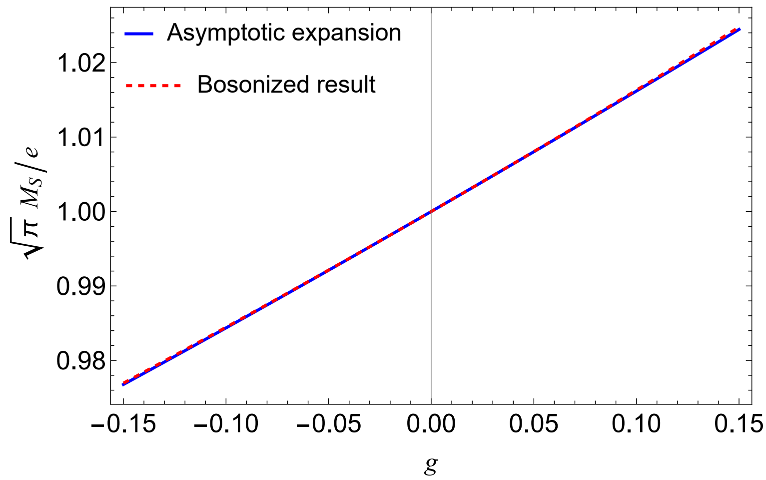

FIG. 1: Plots of equations (15) (red dashed) and (18) (blue line).

[1] C. J. Hamer, J. B. Kogut, D. P. Crewther, and M. M. Mazzolini, Nucl. Phys. B **208**, 413 (1982).

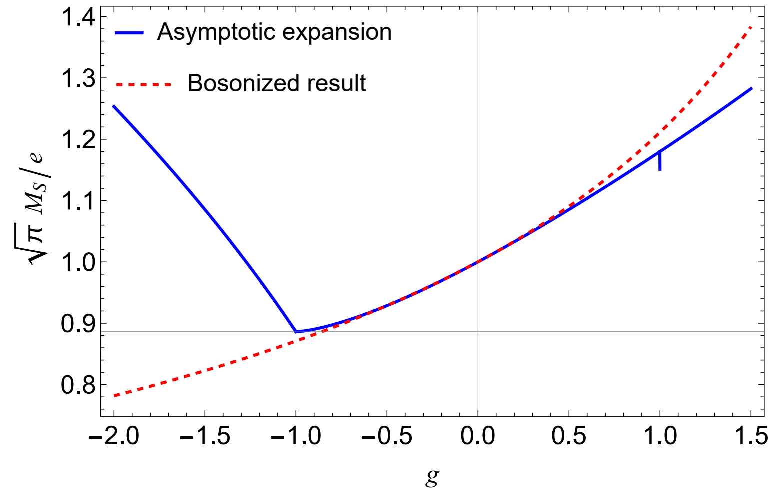

FIG. 2: Extended version of Fig. 1, we can see that the asymptotic expansion works well for the interval $-1 \ll g \ll 1$.