Schwinger Model in a Circle

(Dated: March 28, 2024)

I. LATTICE SCHWINGER MODEL WITH PERIODIC BOUNDARY CONDITIONS

Recall that the Hamiltonian of the Schwinger model on a lattice is

$$H = \sum_{n=1}^{N} [L(n) + \alpha]^2 + x \sum_{n=1}^{N-1} \left[\sigma^+(n) e^{i\theta(n)} \sigma^-(n+1) + \text{c.c.}\right] + \frac{\mu}{2} \sum_{n=1}^{N} (-1)^n \sigma_3(n), \tag{1}$$

where $\mu = \frac{2m_{latt}}{ae^2}$, $m_{latt} = m - \frac{e^2 a}{8}$, $x = \frac{1}{e^2 a^2}$ and a is the lattice spacing. Together with the lattice Gauss law:

$$L(n) - L(n-1) = \frac{1}{2} \left(\sigma_3(n) + (-1)^n\right). \tag{2}$$

The gauge part of the system depends on the choice of boundary conditions: 1) In Open Boundary Conditions (OBC), we can completely eliminate the gauge dependence by fixing an initial electric field and imposing that the incoming field is equal to the outgoing one (which is a consequence of charge conservation), i.e. $E(0)/g = L(0) + \alpha = \alpha = L(N) = E(N)/g$; 2) In Periodic Boundary Conditions (PBC), we have a leftover gauge dependence.

If we focus on PBC, then $L(0) = L(N) = L$. So, our Gauss law reads

$$L(n) = \sum_{k=1}^{n} \frac{1}{2} \left(\sigma_3(k) + (-1)^k\right) + L + \alpha, \tag{3}$$

where α is a fixed background (external) field, and L can live on any of the links of the circle. So, we can eliminate all the gauge dependence up to one θ and its canonically conjugated operator L (recall the commutation relation $[\theta, L] = i$). Let us put such operator on the link that joins the sites N and 1. Therefore, we can rewrite our Hamiltonian for periodic boundary conditions as:

$$\begin{aligned} H_{PBC} = \sum_{n=1}^{N} \left[\sum_{k=1}^{n} \frac{1}{2} \left(\sigma_3(k) + (-1)^k\right) + L + \alpha\right]^2 + x \sum_{n=1}^{N-1} \left[\sigma^+(n)\sigma^-(n+1) + \text{c.c.}\right] + \\ + x \left[\sigma^+(N) e^{i\theta} \sigma^-(1) + \sigma^+(1) e^{-i\theta} \sigma^-(N)\right] + \frac{\mu}{2} \sum_{n=1}^{N} (-1)^n \sigma_3(n). \end{aligned} \tag{4}$$

So, in general, we can consider the following Hamiltonian for the Schwinger model:

$$\begin{aligned} H = \sum_{n=1}^{N} \left[\sum_{k=1}^{n} \frac{1}{2} \left(\sigma_3(k) + (-1)^k\right) + L + \alpha\right]^2 + x \sum_{n=1}^{N-1} \left[\sigma^+(n)\sigma^-(n+1) + \text{c.c.}\right] + \\ + \varepsilon x \left[\sigma^+(N) e^{i\theta} \sigma^-(1) + \sigma^+(1) e^{-i\theta} \sigma^-(N)\right] + \frac{\mu}{2} \sum_{n=1}^{N} (-1)^n \sigma_3(n), \end{aligned} \tag{5}$$

with $\varepsilon = 0$ and $L = 0$ for OBC and $\varepsilon = 1$ for PBC. Notice that,

$$\begin{aligned} H_{PBC} = H_{OBC} + 2 \sum_{n=1}^{N} \left[\sum_{k=1}^{n} \frac{1}{2} \left(\sigma_3(k) + (-1)^k\right) + \alpha\right] L + NL^2 + \\ + \varepsilon x \left[\sigma^+(N) e^{i\theta} \sigma^-(1) + \sigma^+(1) e^{-i\theta} \sigma^-(N)\right]. \end{aligned} \tag{6}$$

Since we have an extra (bosonic) degree of freedom L, then in order to describe the Hilbert space, we consider the basis $|s_1\rangle |s_2\rangle \ldots |s_N\rangle |\ell\rangle$. Where $L|\ell\rangle = \ell |\ell\rangle$ and $e^{\pm i\theta} |\ell\rangle = |\ell \pm 1\rangle$ and $\ell \in \mathbb{Z}$. So, the Hilbert space is infinite

dimensional, that means that we have to truncate our basis up to some fixed L_{max}. Thus $-L_{max} \leq \ell \leq L_{max}$ and there are $(2L_{max}+1)$ states for each chain of spins $|s_1\rangle |s_2\rangle \dots |s_N\rangle$.

From (5) we see that as we increase L, the energy becomes larger, so considering on small values of L_{max} should be enough for the ground and first excited states of the system.

Focusing on the electric field (truncated) Hilbert space, from the commutation relation $[L, \theta] = -i$ one can show that $[L, e^{\pm i\theta}] = \pm e^{\pm i\theta}$. So, let us write what we know so far:

$$\begin{aligned} L\,|L_{max}, \ell\rangle &= \ell\,|L_{max}, \ell\rangle \\ e^{\pm i\theta}\,|L_{max}, \ell\rangle &= |L_{max}, \ell \pm 1\rangle \\ [L, e^{\pm i\theta}] &= \pm e^{\pm i\theta}, \end{aligned} \tag{7}$$

this reminds us about the algebra of $SU(2)$. Recall that the $SU(2)$ algebra irreducible representation for a spin j particle is

$$\begin{aligned} J^z\,|j, m\rangle &= m\,|j, m\rangle \\ J^\pm\,|j, m\rangle &\propto |j, m \pm 1\rangle \\ [J^z, J^\pm] &= \pm J^\pm. \end{aligned} \tag{8}$$

Therefore, we can conclude that the bosonic dof in our compact Schwinger model represents an $SU(2)$ symmetry. With operators L and $e^{\pm i\theta}$ that span the extra Hilbert space[1]. Numerically, we can treat such dof as an additional 'site' with independent spin L_{max} and include it in the OBC Hamiltonian with the additional terms shown in (6) with $\varepsilon = 1$, as depicted in Figure 1. The first thing we can see is that this procedure breaks translational symmetry in the lattice. If we would like to restore it, we could try a different process: for each link include an additional site (as the one shown in Figure 1), so that we have a full chain with $2N$ 'sites' where the odd ones represent the original lattice and the even ones represent the links. Next, identify the link-site $2N$ with the link-site 2 and do the same procedure as before with zero coupling between sites and links, except for one (the bosonic dof). Despite the fact that this preserves translational symmetry, it is more computationally costly in comparison with the previous one.

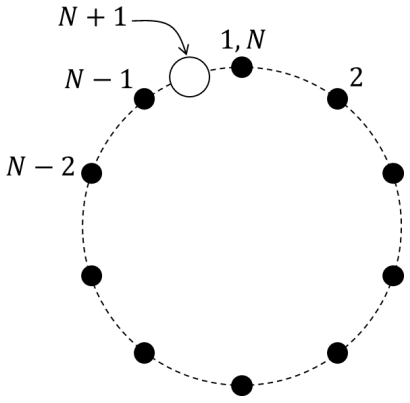

FIG. 1: Schematic picture of the inclusion of the bosonic degree of freedom into the system for numerical purposes.

II. JULIA IMPLEMENTATION

We first set up the tensor indices for our chain with the property that we have a spin 1/2 in the first N sites and spin $Lmax$ in the site $N+1$:

[1] We stress on the fact that this symmetry is a consequence of the truncation of the actual (infinite) Hilbert space of the system.

```
sites = siteinds(n->if n<N+1 "S=1/2" else "Lmax="*string(Lmax) end,N+1;
                 conserve_qns=false)
```

where here the "Lmax=n" with $n \in \mathbb{Z}$ needs to be implemented manually as shown in Appendix A.

Then, we create the OBC Hamiltonian:

```
os = OpSum()      # Initialize a sum of operators for the Hamiltonian

#------------------Hamiltonian for OBC-----------------------
# Calculate Coulomb Hamiltonian (Sz=\sigma/2)
for j in 1:N-1
  os2 = OpSum()
  for k in 1:j
     os2 .+= 1,"Sz", k
     os2 .+= 1/2*(-1)^k, "Id", k
  end
  os2 .+= theta/(2*pi),"Id", j
  # Square the previous sum
  for k in 1:(2*j)+1
     os = y*os2*os2[k] + os
  end
end
# Add the OBC last term (corresponding to L(N)=L+theta/(2*pi))
os += y*(theta/(2*pi))^2, "Id", N

# Mass term in the Hamiltonian
for j in 1:N
  os .+= mu*(-1)^j, "Sz", j
end

# XY Hamiltonian
for j in 1:(N - 1)
  os += x, "S+", j, "S-", j + 1
  os += x, "S-", j, "S+", j + 1
end
```

and include the PBC terms as shown in (6):

```
#--------------PBC terms--------------
#Coulomb additional terms
for j in 1:N-1
  for k in 1:j
      os .+= 2*y*e, "Sz", k, "Sz", N+1
      os .+= y*e*(-1)^k, "Id", k, "Sz", N+1
  end
  os .+= 2*y*e*theta/(2*pi),"Id", j, "Sz", N+1
  os .+= y*e, "Id", j, "Sz", N+1, "Sz", N+1
end
# Additional terms of (L+theta/(2*pi))^2
os .+= 2*y*e*theta/(2*pi),"Id", N, "Sz", N+1
os .+= y*e, "Id", N, "Sz", N+1, "Sz", N+1

 # XY additional terms
os .+= x*e,"S+", N, "S+", N+1, "S-", 1
os .+= x*e,"S-", N, "S-", N+1, "S+", 1
```

where e= 0 means OBC and e= 1 corresponds to PBC.

III. NUMERICAL RESULTS

A. Massless Case

Considering the massless case: $\mu = 1/4$, we have that the model is exactly solvable and we can obtain the spectrum of excitations of the system that are governed by zero-momentum multiparticle states (with energies as multiples of the boson mass $M = e/\sqrt{\pi}$) and single particle non-zero momentum states (with energies given by $\mathcal{E}_k^2 = M^2 + \left(\frac{2\pi k}{L}\right)^2$, with $k = 0, 1, \ldots, L-1$) [1]. Thus, by calculating the first energy levels of the spin Hamiltonian in (6), we can obtain some of such excitations and compare with the exact results. We obtain the first three energy levels for both $\theta = 0$ and $\theta = \pi$, since for the massless case, the system is symmetric with respect to the two values of the background field. So, the difference between them give us the precision of our calculations. These results are shown in the following tables, using DMRG in *Julia* with bond dimension of 2000.

$N = 4$, $x = 100$, sweeps= 25

Parameters $(m/e = 0)$	$E_0(S_z = 0)$	$E_1(S_z = 0)$	$E_2(S_z = 0)$	$E_3(S_z = 0)^*$
$\theta = \pi$, $L_{max} = 1$	−272.08499708443753	−245.2734897572119	−205.3370269505082	−191.70150211610235
$\theta = 0$, $L_{max} = 1$	−272.0849970844373	−245.27348975721176	−205.34333346269497	−190.59649190086753
$\theta = \pi$, $L_{max} = 3$	−276.27618974358575	−264.3203188059014	−251.94542767344612	−238.49082044836425
$\theta = 0$, $L_{max} = 3$	−276.27618974358586	−264.3203188059016	−251.94542767344586	−238.49082044836445
$\theta = \pi$, $L_{max} = 5$	−276.28737100388815	−264.45812712510985	−252.6920505027035	−240.98894684693923
$\theta = 0$, $L_{max} = 5$	−276.2873710038878	−264.4581271251098	−252.6920505027036	−240.9889468469388

TABLE I: *The values of E_3 for $L_{max} = 1$ have different spins: $S_z = -1$ for $\theta = \pi$ and $\theta = 0$.

$N = 8$, $x = 100$, sweeps= 40

Parameters $(m/e = 0)$	$E_0(S_z = 0)$	$E_1(S_z = 0)$	$E_2(S_z = 0)$	$E_3(S_z = 0)^*$
$\theta = \pi$, $L_{max} = 1$	−513.8023839716004	−496.41065925963596	−472.7707942884465	−473.7314083709304
$\theta = 0$, $L_{max} = 1$	−513.8023839716004	−496.41065925963653	−472.77079410146257	−474.2313254122319
$\theta = \pi$, $L_{max} = 3$	−515.3967418689417	−503.99688273271335	−492.6222185508734	−481.24494798698873
$\theta = 0$, $L_{max} = 3$	−515.3967418689429	−503.99688273271397	−492.6222185508733	−481.24494798698845
$\theta = \pi$, $L_{max} = 5$	−515.3967742009228	−503.9976169739953	−492.6299045630504	−481.29376872649766
$\theta = 0$, $L_{max} = 5$	−515.3967742009229	−503.99761697399464	−492.62990456305056	−481.29376872571936

TABLE II: *The values of E_3 for $L_{max} = 1$ have different spins: $S_z = -1$ for $\theta = \pi$ and $S_z = +1$ for $\theta = 0$.

$N = 12$, $x = 100$, sweeps= 100

Parameters $(m/e = 0)$	$E_0(S_z = 0)$	$E_1(S_z = 0)$	$E_2(S_z = 0)^*$	$E_3(S_z = -1)$
$\theta = \pi$, $L_{max} = 1$	−763.5955481037167	−748.7806301940523	−736.3162138010072	−734.5088186563228
$\theta = 0$, $L_{max} = 1$	−763.5955481037158	−748.7806301940532	−736.8162138010066	−734.0088186563225
$\theta = \pi$, $L_{max} = 3$	−764.5475418128043	−753.2198435484826	−741.9129370112697	−737.1022857837995
$\theta = 0$, $L_{max} = 3$	−764.5475418128027	−753.2198435484842	−741.9129370112702	−737.1002525357076
$\theta = \pi$, $L_{max} = 5$	−764.5475421941898	−753.2198545342728	−741.9130873337047	−737.1019886840583
$\theta = 0$, $L_{max} = 5$	−764.5475421941892	−753.2198545342711	−741.9130873337047	−737.1089486021432

TABLE III: *The values of E_2 for $L_{max} = 1$ have different spins: $S_z = 1$ for $\theta = \pi$ and $\theta = 0$.

As shown in Figure 2, we can see that the gaps approach to the spectrum of the continuum theory[2] with very

[2] The value of $Le = \frac{N-2}{2}$ is just a guess, some deeper analysis should be done.

$N = 16$, $x = 100$, sweeps= 100

Parameters $(m/e = 0)$	$E_0(S_z = 0)$	$E_1(S_z = 0)$	$E_2(S_z = +1)^*$	$E_3(S_z = -1)^*$
$\theta = \pi$, $L_{max} = 1$	-1015.2283726004712	-1001.5222533664512	-994.1486466849917	-992.1461633347575
$\theta = 0$, $L_{max} = 1$	-1015.2283726004717	-1001.5222533664511	-994.6486466849914	-991.646163334758
$\theta = \pi$, $L_{max} = 3$	-1015.9148883308482	-1004.6104109390375	-995.2247510640028	-994.7313221032641
$\theta = 0$, $L_{max} = 3$	-1015.9148883308485	-1004.6104109390371	-995.230130497358	-994.7259433730328
$\theta = \pi$, $L_{max} = 5$	-1015.9148883442372	-1004.6104113641009	-995.2259124316716	-994.730164973228
$\theta = 0$, $L_{max} = 5$	-1015.9148883442348	-1004.6104113640994	-995.2301102669293	-994.7259671330488

TABLE IV: *The values of $\theta = \pi$ and $L_{max} = 1$ have spin $S_z = -1$ for E_2 and $S_z = +1$ for E_3.

$N = 20$, $x = 100$, sweeps= 100

Parameters $(m/e = 0)$	$E_0(S_z = 0)$	$E_1(S_z = 0)$	$E_2(S_z = +1)^*$	$E_3(S_z = -1)^*$
$\theta = \pi$, $L_{max} = 1$	-1267.5748046788178	-1254.4593014875213	-1250.0854954073648	-1247.9579079869247
$\theta = 0$, $L_{max} = 1$	-1267.5748046788174	-1254.4593014875231	-1250.5854954073639	-1247.4579079869275
$\theta = \pi$, $L_{max} = 3$	-1268.124444809774	-1256.8299624736703	-1251.051537762345	-1250.55892189605
$\theta = 0$, $L_{max} = 3$	-1268.124444809776	-1256.8299624736726	-1251.0585351105424	-1250.5519247595346
$\theta = \pi$, $L_{max} = 5$	-1268.124444810873	-1256.8299625095424	-1251.0519187267603	-1250.558541844667
$\theta = 0$, $L_{max} = 5$	-1268.1244448108732	-1256.8299625095378	-1251.0585153206255	-1250.5519452432127

TABLE V: *The values of $\theta = \pi$ and $L_{max} = 1$ have spin $S_z = -1$ for E_2 and $S_z = +1$ for E_3.

small number of sites (recall that the jth gap is $\Delta_j = \frac{E_j - E_0}{2\sqrt{x}}$). So, extrapolating[3] $N \to \infty$, we obtain $M_1/e = 0.5641799890656634$ which is $\sim 9 \cdot 10^{-6}$ away from its exact value $\frac{1}{\sqrt{\pi}}$. Similarly, $M_2/e = 1.1292614285641598$ and $M_3/e = 1.6453793396729026$ which are $\sim 9 \cdot 10^{-4}$ and $\sim 5 \cdot 10^{-2}$ away from their exact values $\frac{2}{\sqrt{\pi}}$ and $\frac{3}{\sqrt{\pi}}$, respectively (as shown in Figure 3). The largest precision of our calculations is obtained by comparing results using $\theta = 0$ and $\theta = \pi$ and is of order 10^{-11}. Note that there are two very close states in Figure 2 for $N = 16$, 20 corresponding to the $S = \pm 1$ states. Since we are in the massless case, the system is invariant under a $\mathbb{Z}_2$ charge conjugation $\mathcal{C}$ symmetry, therefore both states are degenerate in the continuum limit (the small difference may be a consequence of finite lattice effects).

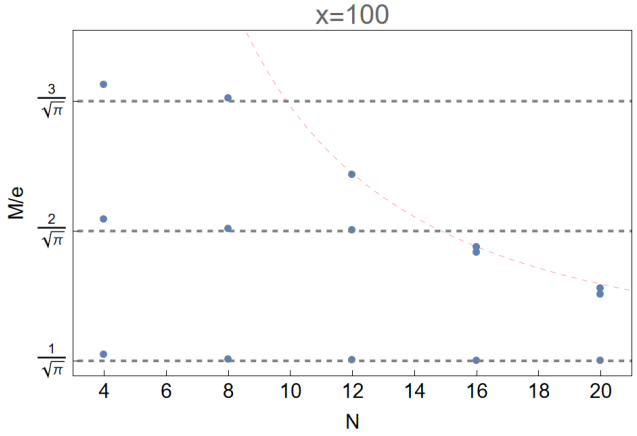

FIG. 2: First three energy gaps of the spectrum of the system for $x = 100$, $L_{max} = 5$ and $\theta = 0$. The red dashed line corresponds to the curve $\mathcal{E}_1 = \sqrt{M^2 + \left(\frac{2\pi}{L}\right)^2}$ with $L = \frac{N-2}{2e}$.

[3] This is not quite the continuum limit, since it still depends on x. For the real continuum value, after sending $N \to \infty$, we need to extrapolate $x \to \infty$ ($a \to 0$).

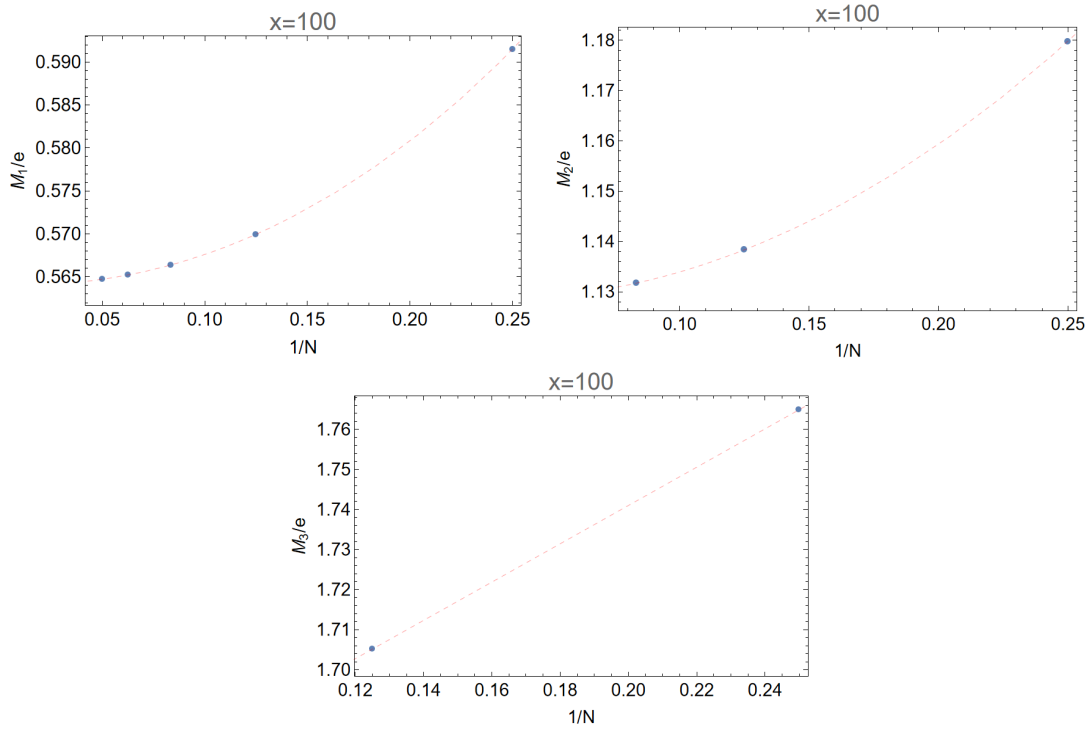

FIG. 3: Extrapolation of the values of $\Delta_1,\ \Delta_2,\ \Delta_3$ for $N \to \infty$. The red dashed lines corresponds to the fits $M_1 + \frac{a_1}{N} + \frac{a_2}{N^2} + \frac{a_3}{N^3} + \frac{a_4}{N^4}$ for Δ_1, $M_2 + \frac{b_1}{N} + \frac{b_2}{N^2}$ for Δ_2, and $M_3 + \frac{c_1}{N}$ for Δ_3.

B. Massive Case

For this case, we will focus on finding the critical point of the model m_c and we can use one of two methods (or both): 1) Finite-size scaling: we find the values of the ratio [2]

$$R(N, x, m) = \frac{N\ M(N, x, m)}{(N-2)M(N-2, x, m)} \tag{9}$$

where $M(N, x, m)$ is the mass gap of the system with N sites and parameters x and m. The idea is to determine the critical masses $m_c(N, x)$ by making use of the crossing method, i.e. looking for $R(N, x, m_c) = 1$ and then extrapolate to $N \to \infty$. Finally, the continuum limit critical mass is obtained by $m_c = \lim_{x\to\infty} m_c(\infty, x)$.

2) Entanglement entropy: By considering the invariant RG ratio [3]:

$$Q(\ell, \ell') = 3\left[\frac{S_A^L(\ell) - S_A^L(\ell')}{\log\sin(\pi\ell/L) - \log\sin(\pi\ell'/L)}\right], \tag{10}$$

where $S_A^L(\ell)$ is the entanglement entropy between a subinterval of length ℓ with its complement in a system with PBC of size L. If we are at the critical point, it has a well-known form [4]:

$$S_A(\ell) = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi\ell}{L}\right) + c_1', \tag{11}$$

with c as the central charge of the CFT and $\tilde{c}_1$ is called the boundary entropy. Therefore,

$$\lim_{N\to\infty} Q(\ell, \ell') = c(x). \tag{12}$$

While, away from m_c, for $L \gg \xi$ (ξ being the correlation length) we have that $S_A \sim \log \xi$ and $Q(\ell, \ell') = 0$. Therefore, equation (10) implies that, if we plot the function $Q(m)$ for different N, then there will be a maximum which indicates the location of $m_c(N, x)$. As before, to obtain the continuum limit results, we should calculate $m_c = \lim_{x\to\infty} m_c(\infty, x)$ and $c = \lim_{x\to\infty} c(x)$. To find $m_c(N, x)$ more precisely, we can use the intersection method again, by looking for when the ratio $\frac{Q(L/2,L/4)}{Q((L-2)/2,(L-2)/4)} = 1$. The advantage of 2) over 1) is that we do not need to calculate excited energy levels, since the ground state is enough for obtaining the entanglement entropy via MPS.

Note: It is important to know that when implementing the Schwinger model code, we have to be aware that our parameters are such that $x < N$, this because the limits $N \to \infty$ and $x \to \infty$ do not commute. Therefore, the bigger the number of lattice sites, the smaller values of the lattice spacing $a = 1/(ex)$ we can consider.

Appendix A: Julia code for higher spins on each site

Here is a code example of how to implement sites of the lattice in *Julia* with $L_{max} = 2$. Generalizations can be done with the same idea (notice the difference in normalization in the $S^{\pm}$ matrices compared to the usual spin representation).

```
    """
    space(::SiteType"Lmax=2";
          conserve_qns = false,
          conserve_sz = conserve_qns,
          qnname_sz = "Sz")

Create the Hilbert space for a site of type "Lmax=2".
Optionally specify the conserved symmetries and their quantum number labels."""

function ITensors.space(
  ::SiteType"Lmax=2"; conserve_qns=false, conserve_sz=conserve_qns,
  qnname_sz="Sz"
)
  if conserve_sz
    return [QN(qnname_sz, +4) => 1, QN(qnname_sz, +2) => 1,
    QN(qnname_sz, 0) => 1, QN(qnname_sz, -2) => 1, QN(qnname_sz, -4) => 1]
  end
  return 5
end

ITensors.val(::ValName"+2", ::SiteType"Lmax=2") = 1
ITensors.val(::ValName"+1", ::SiteType"Lmax=2") = 2
ITensors.val(::ValName"0", ::SiteType"Lmax=2") = 3
ITensors.val(::ValName"-1", ::SiteType"Lmax=2") = 4
ITensors.val(::ValName"-2", ::SiteType"Lmax=2") = 5

ITensors.state(::StateName"+2", ::SiteType"Lmax=2") = [1.0, 0.0, 0.0, 0.0, 0.0]
ITensors.state(::StateName"+1", ::SiteType"Lmax=2") = [0.0, 1.0, 0.0, 0.0, 0.0]
ITensors.state(::StateName"0", ::SiteType"Lmax=2") = [0.0, 0.0, 1.0, 0.0, 0.0]
ITensors.state(::StateName"-1", ::SiteType"Lmax=2") = [0.0, 0.0, 0.0, 1.0, 0.0]
ITensors.state(::StateName"-2", ::SiteType"Lmax=2") = [0.0, 0.0, 0.0, 0.0, 1.0]

function ITensors.op!(Op::ITensor, ::OpName"Sz", ::SiteType"Lmax=2", s::Index)
  Op[s' => 1, s => 1] = +2.0
  Op[s' => 2, s => 2] = +1.0
  Op[s' => 3, s => 3] = 0.0
  Op[s' => 4, s => 4] = -1.0
  return Op[s' => 5, s => 5] = -2.0
end
```

```
function ITensors.op!(Op::ITensor, ::OpName"S+", ::SiteType"Lmax=2", s::Index)
  Op[s' => 2, s => 3] = +1.0
  Op[s' => 3, s => 4] = +1.0
  Op[s' => 4, s => 5] = +1.0
  return Op[s' => 1, s => 2] = +1.0

function ITensors.op!(Op::ITensor, ::OpName"S-", ::SiteType"Lmax=2", s::Index)
  Op[s' => 3, s => 2] = +1.0
  Op[s' => 4, s => 3] = +1.0
  Op[s' => 5, s => 4] = +1.0
  return Op[s' => 2, s => 1] = +1.0
end

ITensors.space(::SiteType"SpinTwoMod"; kwargs...) =
             space(SiteType("Lmax=2"); kwargs...)

ITensors.state(name::StateName, ::SiteType"SpinTwoMod") =
             state(name, SiteType("Lmax=2"))
ITensors.val(name::ValName, ::SiteType"SpinTwoMod") =
             val(name, SiteType("Lmax=2"))

function ITensors.op!(Op::ITensor, o::OpName, ::SiteType"SpinTwoMod", s::Index)
  return op!(Op, o, SiteType("Lmax=2"), s)
end
```

---

[1] R. Dempsey, I. R. Klebanov, S. S. Pufu, and B. Zan, "Discrete chiral symmetry and mass shift in lattice hamiltonian approach to schwinger model," 2023.

[2] C. Hamer and M. N. Barbert, "Finite-size scaling in hamiltonian field theory," in *Current Physics–Sources and Comments*, vol. 2, pp. 258–263, Elsevier, 1988.

[3] M. Campostrini, A. Pelissetto, and E. Vicari, "Finite-size scaling at quantum transitions," *Physical Review B*, vol. 89, Mar. 2014.

[4] P. Calabrese and J. Cardy, "Entanglement entropy and quantum field theory," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2004, p. P06002, June 2004.