Schwinger Model Comments

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This series of notes are compiled from: Schwinger '62 (Gauge Invariance and Mass), Kogut and Susskind '74 (Hamiltonian formulation of Wilson's lattice gauge theories), Banks, Susskind, and Kogut, '75 (Strong Coupling calculations of lattice gauge theories: $(1+1)-$dimensional exercises), Coleman, Jackiw, and Susskind '75 (Charge Shielding and Quark Confinement in the Massive Schwinger Model), Coleman '76 (More about the Massive Schwinger Model), Hamer, Weihong, and Oitmaa, '97 (Series expansions for the massive Schwinger model in Hamiltonian lattice theory), Byrnes '03 (Density matrix renormalization group : a new approach to lattice gauge theory), Tong '18 (QFT on the line).

1 Quantum Field Theory in 2 dimensions

1.1 Electromagnetism in (1+1)-dimensions

In (1+1) dimensions, we consider the metric $g_{\mu\nu} = \mathrm{diag}(1, -1)$. Here, the field strength has only one component $F_{\mu\nu} = F_{01} = -F_{10}$. And the action is

$$S = \int d^2x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu \right], \tag{1}$$

where j^μ is the current of the charged matter coupled to the gauge field A_μ and the electric field is just $E(x) = F^{01}$.

The classical equations of motion (EoM) are

$$\partial_\nu F^{\mu\nu} = j^\mu \tag{2}$$

Note that in the absence of interactions, Gauss' law imply that $\partial_x E(x) = 0$, meaning that the electric field is constant. That is, there are no electromagnetic wave solutions in (1+1), this because we do not have propagating degrees of freedom (no transverse modes).

Let us consider adding a point charge (with charge qe) at the origin, so that the EoM are

$$\begin{aligned} \partial_x E(x) &= qe\delta(x) \\ E(x) &= qe\Theta(x) + F, \end{aligned} \tag{3}$$

with F being a background electric field, fixed at spatial infinity. We can see that the electric field emitted by a point charge is constant and the corresponding energy of putting such point charge is

$$E = \int dx \frac{E(x)^2}{2} = \text{infinite} \tag{4}$$

This is a problem, and tells us that point charges cannot be physical states of our system, as depicted in Figure 1 a). Now consider placing 2 point charges with opposite charge and separated by a distance L, then (assuming a vanishing background field)

$$\begin{aligned}
\partial_x E(x) &= qe\left[\delta\left(x+\frac{L}{2}\right) - \delta\left(x-\frac{L}{2}\right)\right] \\
E(x) &= qe\left[\Theta\left(x+\frac{L}{2}\right) - \Theta\left(x-\frac{L}{2}\right)\right] \\
E(x) &= \begin{cases} qe & , \quad x \in \left(-\frac{L}{2}, \frac{L}{2}\right) \\ 0 & , \quad \text{otherwise} \end{cases}
\end{aligned} \tag{5}$$

whose corresponding energy is

$$E = \int dx \frac{E(x)^2}{2} = \frac{q^2 e^2}{2} L. \tag{6}$$

The energy is finite and the physical states are then neutral. Besides, the energy grows linearly with the separation of the charges. So, in (1+1) electric charges are classically confined. The possible reason for this is that the electric field is forced to form a "flux tube" because it has nowhere else to go (Figure 1 b)). So, the spectrum will be a tower of neutral meson-like states, each containing particle-antiparticle pairs.

Notice that, naively, the string L breaks when $E_{\text{string}} = \frac{q^2 e^2 L}{2} > 2m_{\text{particle}} = E_{\text{treshold}}$, that is approximately when

$$L \geq \frac{4m_{\text{particle}}}{q^2 e^2}.$$

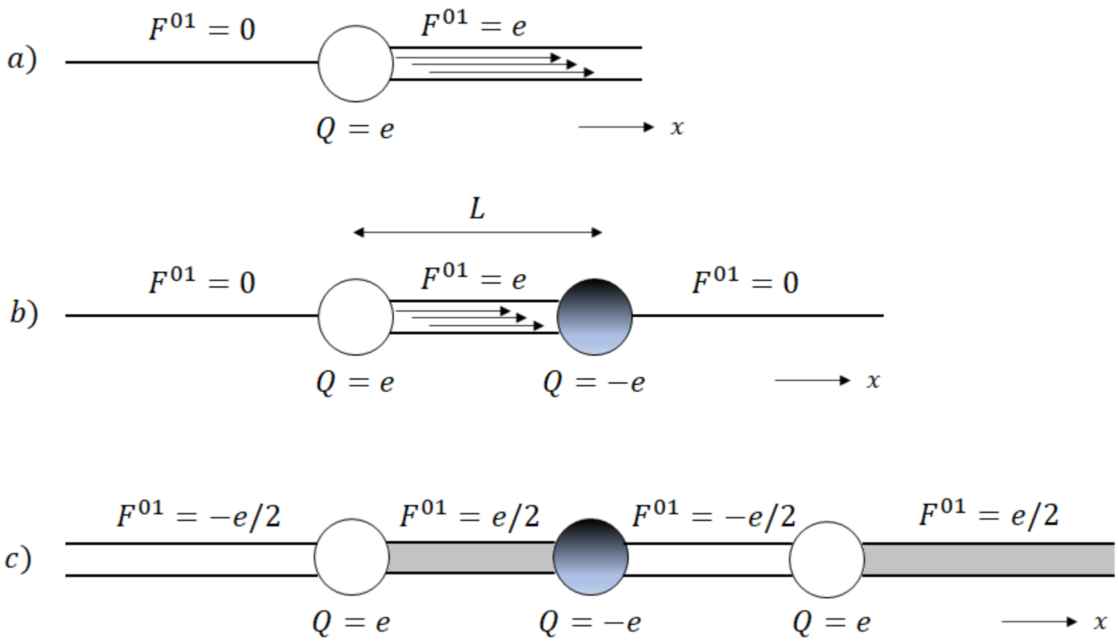

Figure 1: Addition of point charges to the (1+1)-D system. a) Will cost an infinite amount of energy. b) An electric flux tube is formed, showing confinement between particles. c) Configuration with a background field in which particles are no longer confined and free to move.

1.2 Background Electric Field

In principle, we could argue that F is zero at the spatial infinities, as in (3+1) dimensions. But let us study this more carefully. Suppose that in (3+1)-D there IS a background field F, then the vacuum will suffer a "dielectric breakdown", namely it will create an electron-positron pair that will reduce the overall constant field between the two particles. The it is energetically more favorable that they travel to infinity reducing the whole background field. This process of $e\bar{e}$ production will be repeated until $F \to 0$. So, F would be cancelled by vacuum pair production.

However, in (1+1)-D, the pair production is slightly different. We have seen that putting a particle-antiparticle pair in an electromagnetic system increases (reduces) the background field by qe ($-qe$) if the positive charge is at the left (right) correspondingly. Therefore, taking $q = 1$, the change in energy by this production is

$$\Delta E = \frac{1}{2} \int dx \left(E(x)^2 - F^2 \right) = \frac{L}{2} \left[(F \pm e)^2 - F^2 \right] = \frac{eL}{2} \left[\pm 2F + e \right]. \tag{7}$$

Thus, it is not favorable for the vacuum to produce a pair if $|F| \leq e/2$ because $\Delta E \geq 0$.

If $|F| > e/2$, then pairs will be produced until $|F| \leq e/2$. This tells us that the background electric field has period e. Now let

$$\theta \equiv \frac{2\pi F}{e}, \tag{8}$$

so that $\theta \in [-\pi, \pi]$. We call this 'Theta Angle' as the background field with period 2π.

So, in general, we include this term in the action as a total derivative (for not affecting the EoM), viz

$$S = \int d^2x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e \frac{\theta}{2\pi} \frac{\epsilon^{\mu\nu} F_{\mu\nu}}{2} + A_\mu j^\mu \right], \tag{9}$$

where $\epsilon^{\mu\nu}$ is the 2D Levi-Civita tensor and $\epsilon^{01} = 1 = -\epsilon^{10}$.

Let us consider our system with a background field at $x = -\infty$ of $F = -e/2$ or $\theta = -\pi$, then the presence of a particle positively charged $Q = +e$ will change the background field to $F = e/2$ or $\theta = \pi$ at $x = \infty$. Both values have the same magnitude and, interestingly, particles are now free to move along the line with no energy cost ($\Delta E = \frac{ex}{2}(2F + e) = 0$). That is, particles are not confined anymore. We can see this with more particles, as shown in Figure 1 c), string tensions cancel on either side and alternating particle/antiparticles no longer feel the distance force.

1.3 Fermions in 2D

Consider the Clifford algebra[1] in (1+1)-D:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10}$$

The corresponding action is

$$\begin{aligned} S &= \int d^2x \left[i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \right] \\ &= \int d^2x \left[i\psi_+^\dagger \partial_- \psi_+ + i\psi_-^\dagger \partial_+ \psi_- - m(\psi_-^\dagger \psi_+ + \psi_+^\dagger \psi_-) \right], \end{aligned} \tag{11}$$

where we have used the light cone coordinates $x^\pm = t \pm x$, $\partial_\pm = \partial_t \pm \partial_x$ and $\psi = (\psi_+, \psi_-)^T$. Note that ψ is an eigenvector of the matrix $\gamma^3 = \gamma^0\gamma^1 = -\sigma^3$.

We have a $U(1)$ symmetry that leads to a conserved vector current:

$$j_V^\mu = \bar{\psi}\gamma^\mu\psi, \tag{12}$$

and, for the massless case, we also have an axial conserved current:

$$j_A^\mu = \bar{\psi}\gamma^\mu\gamma^3\psi. \tag{13}$$

The equations of motion for the massless case are

$$\begin{cases} \partial_-\psi_+ = 0 & \Longrightarrow \quad \psi_+ = \psi_+(x^+) \quad \text{“left-moving” fermion} \\ \partial_+\psi_- = 0 & \Longrightarrow \quad \psi_- = \psi_-(x^-) \quad \text{“right-moving” fermion} \end{cases} \tag{14}$$

Now, we proceed to quantize the theory by noticing that the “right-moving” fermion will have momentum $p > 0$ and the “left-moving” fermion will have momentum $p < 0$:

[1] In this section we will use the Weyl representation for fermions, this will be easier when we will talk about quantization and bosonization.

$$\psi_-(x) = \int_0^\infty \frac{dp}{2\pi}\left(b_{-,p}e^{ipx} + c^\dagger_{-,p}e^{-ipx}\right) \tag{15}$$

$$\psi_+(x) = \int_{-\infty}^0 \frac{dp}{2\pi}\left(b_{+,p}e^{ipx} + c^\dagger_{+,p}e^{-ipx}\right), \tag{16}$$

with

$$\left\{b_{\pm,p}, b^\dagger_{\pm,q}\right\} = \left\{c_{\pm,p}, c^\dagger_{\pm,q}\right\} = 2\pi\delta(p-q), \quad b_{\pm,p}\left|0\right\rangle = c_{\pm,p}\left|0\right\rangle = 0 \text{ (vacuum)}. \tag{17}$$

To deal with the UV divergences, we regulate the integrals as follows:

$$\psi_-(x) = \int_0^\infty \frac{dp}{2\pi}\left(b_{-,p}e^{ipx} + c^\dagger_{-,p}e^{-ipx}\right)e^{-|p|/(2\Lambda)} \tag{18}$$

$$\psi_+(x) = \int_{-\infty}^0 \frac{dp}{2\pi}\left(b_{+,p}e^{ipx} + c^\dagger_{+,p}e^{-ipx}\right)e^{-|p|/(2\Lambda)}, \tag{19}$$

here Λ is the UV cut-off scale. Therefore, the two-point function is

$$\begin{aligned}
\langle\psi_-(x)\psi^\dagger_-(y)\rangle &= \int_0^\infty \frac{dpdq}{(2\pi)^2}\langle b_{-,p}b^\dagger_{-,q}\rangle\, e^{ipx-ipy}e^{-(|p|+|q|)/(2\Lambda)} \\
&= \int_0^\infty \frac{dp}{2\pi}e^{ip(x-y)}e^{-|p|/\Lambda}, \quad |p| = p \\
&= -\frac{1}{2\pi}\frac{1}{i(x-y)-\frac{1}{\Lambda}} \\
&= \frac{i}{2\pi}\frac{1}{(x-y)+i\varepsilon}, \quad \varepsilon = \frac{1}{\Lambda}.
\end{aligned} \tag{20}$$

Similarly,

$$\begin{aligned}
\langle\psi_+(x)\psi^\dagger_+(y)\rangle &= \int_{-\infty}^0 \frac{dpdq}{(2\pi)^2}\langle b_{+,p}b^\dagger_{+,q}\rangle\, e^{ipx-ipy}e^{-(|p|+|q|)/(2\Lambda)} \\
&= \int_{-\infty}^0 \frac{dp}{2\pi}e^{ip(x-y)}e^{-|p|/\Lambda}, \quad |p| = -p \\
&= \frac{1}{2\pi}\frac{1}{i(x-y)+\frac{1}{\Lambda}} \\
&= -\frac{i}{2\pi}\frac{1}{(x-y)-i\varepsilon}, \quad \varepsilon = \frac{1}{\Lambda}.
\end{aligned} \tag{21}$$

1.4 Bosonization

Let's consider a theory of a massless periodic scalar field, with radius β:

$$S_\phi = \int d^2x \frac{\beta^2}{2}(\partial_\mu\phi)^2, \tag{22}$$

the reason of why we do not reabsorb β into ϕ is because we will change the periodicity of the field. How about the form of interactions? Terms like ϕ^2, ϕ^4 do not respect periodicity, but terms such as $\cos(\phi)$ or $\sin(\phi)$ do.

It is easy to see that the action is invariant under $\phi \to \phi + \text{constant}$. Thus, there is a conserved current associated to translations (shifts)

$$j^{\mu}_{shift} = \beta^2(\partial^{\mu}\phi), \tag{23}$$

whose charge $Q_{shift} = \int_0^{2\pi R} dx \beta^2 \partial_t \phi$.

We also have an additional current:

$$j^{\mu}_{wind} = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_{\nu}\phi, \tag{24}$$

this is conserved by construction and its charge $Q_{wind} = \int_0^{2\pi R} dx \frac{1}{2\pi}\partial_x\phi$ =number of times the field winds around its range as we go around the spatial circle.

The fact that we have two $U(1)$ symmetries reminds us of the vector and axial ones in the massless fermionic case.

For the canonical quantization of the boson, let's ignore for now its compactness, so

$$\phi(x) = \frac{1}{\beta}\int \frac{dp}{2\pi}\frac{1}{\sqrt{2|p|}}\left(a_p e^{ipx} + a_p^{\dagger}e^{-ipx}\right)e^{-|p|/(2\Lambda)}, \tag{25}$$

and since classically $\pi(x) = \beta^2\dot{\phi}$, then

$$\pi(x) = -i\beta\int \frac{dp}{2\pi}\sqrt{\frac{|p|}{2}}\left(a_p e^{ipx} - a_p^{\dagger}e^{-ipx}\right)e^{-|p|/(2\Lambda)}, \tag{26}$$

together with the commutation relations

$$[a_p, a_q^{\dagger}] = (2\pi)\delta(p-q) \implies [\phi(x), \pi(y)] = i\delta(x-y). \tag{27}$$

To make a complete analogy, we need to write the scalar field as a combination of chiral bosons, in order to do so, consider the operators:

$$\phi_{\pm} = \frac{1}{2}\left[\phi(x) \pm \frac{1}{\beta^2}\int_{-\infty}^{x} dx'\pi(x')\right], \tag{28}$$

we can see that $\phi_{\pm}$ are non-local objects, therefore let's verify if they are well-defined:

$$\begin{aligned}\phi_- &= \frac{1}{2\beta}\int\frac{dp}{2\pi}\left[\frac{1}{\sqrt{2|p|}}\left(a_p e^{ipx} + a_p^{\dagger}e^{-ipx}\right)e^{-|p|/(2\Lambda)} + i\frac{1}{ip}\sqrt{\frac{|p|}{2}}\left(a_p e^{ipx} + a_p^{\dagger}e^{-ipx}\right)e^{-|p|/(2\Lambda)}\right]\\ &= \frac{1}{2\beta}\left[\int_0^{\infty}\frac{dp}{2\pi}\frac{1}{\sqrt{2|p|}}\left(1 + \frac{|p|}{p}\right)\left(a_p e^{ipx} + a_p^{\dagger}e^{-ipx}\right)e^{-|p|/(2\Lambda)}\right]\\ &= \frac{1}{2\beta}\left[\int_0^{\infty}\frac{dp}{2\pi}\sqrt{\frac{2}{p}}\left(a_p e^{ipx} + a_p^{\dagger}e^{-ipx}\right)e^{-p/(2\Lambda)}\right], \quad p > 0.\end{aligned} \tag{29}$$

Similarly,

$$\phi_+ = \frac{1}{2\beta}\left[\int_{-\infty}^{0}\frac{dp}{2\pi}\sqrt{\frac{2}{|p|}}\left(a_p e^{ipx} + a_p^{\dagger}e^{-ipx}\right)e^{-|p|/(2\Lambda)}\right], \quad p < 0. \tag{30}$$

Now, the 2-point function has the (regularized) form:

$$G_\pm(x,y) = \langle \phi_\pm(x)\phi_\pm(y)\rangle - \underbrace{\langle \phi_\pm(0)^2\rangle}_{\text{to avoid UV divergences}} \quad . \tag{31}$$

So,

$$\begin{aligned}
G_-(x,y) &= \frac{1}{4\beta^2}\int_0^\infty \frac{dpdq}{(2\pi)^2}\frac{2}{\sqrt{pq}}\underbrace{\langle a_p a_q^\dagger\rangle}_{2\pi\delta(p-q)}(e^{ipx-iqx}-1)e^{\frac{-p-q}{2\Lambda}} \\
&= \frac{1}{4\beta^2}\int_0^\infty \frac{dp}{2\pi}\frac{2}{p}(e^{ip(x-y)}-1)e^{-p/\Lambda} \\
&= \frac{1}{4\pi\beta^2}\int_0^\infty dp\int_{\frac{i}{\Lambda}}^{x-y+i/\Lambda} dz\ ie^{izp} \\
&= \frac{1}{4\pi\beta^2}\int_{\frac{i}{\Lambda}}^{x-y+i/\Lambda} dz\ i\frac{1}{iz}\left(\lim_{p\to\infty} e^{izp}-1\right) \\
&= \frac{1}{4\pi\beta^2}\log\left(\frac{i/\Lambda}{x-y+i/\Lambda}\right) \\
&= \frac{1}{4\pi\beta^2}\log\left(\frac{\varepsilon}{\varepsilon-i(x-y)}\right), \quad \varepsilon = \frac{1}{\Lambda}.
\end{aligned} \tag{32}$$

Similarly,

$$G_+(x,y) = \frac{1}{4\pi\beta^2}\log\left(\frac{\varepsilon}{\varepsilon+i(x-y)}\right), \quad \varepsilon = \frac{1}{\Lambda}. \tag{33}$$

Notice that $G_\pm(x,y) = G_\pm(x-y)$ and $G_\pm(0) = 0$.

Since ϕ itself is not a good operator because it is not single valued, then it is better to work with either $\partial\phi$ or $e^{i\phi}$, therefore the (normal-ordered) quantity

$$\begin{aligned}
\langle : e^{i\phi_-(x)} :: e^{-i\phi_-(y)} :\rangle &= \langle : e^{i\phi_-(x)-i\phi_-(y)} :\rangle\, e^{\langle[\phi_-(x),\phi_-(y)]\rangle} \\
&= e^{G_-(x-y)} \\
&= \left(\frac{\varepsilon}{\varepsilon-i(x-y)}\right)^{\frac{1}{4\pi\beta^2}},
\end{aligned} \tag{34}$$

where in the first line we used Baker-Campbell-Hausdorff (BCH) formula. Analogously,

$$\langle : e^{i\phi_+(x)} :: e^{-i\phi_+(y)} :\rangle = \left(\frac{\varepsilon}{\varepsilon+i(x-y)}\right)^{\frac{1}{4\pi\beta^2}}. \tag{35}$$

If we make the following identifications:

$$\begin{aligned}
\psi_-(x) &\longrightarrow \frac{1}{\sqrt{2\pi\varepsilon}} : e^{i\phi_-(x)} : \\
\psi_+(x) &\longrightarrow \frac{1}{\sqrt{2\pi\varepsilon}} : e^{-i\phi_+(x)} : \\
\beta &\longrightarrow \sqrt{\frac{1}{4\pi}}
\end{aligned} \tag{36}$$

then our Green's functions coincides with those of the massless fermionic case. Also, the composite operator is identified as

$$
\begin{aligned}
\bar{\psi}\psi &= \psi_-^\dagger(x)\psi_+(x) + \psi_+^\dagger(x)\psi_-(x) \\
&\longleftrightarrow \frac{1}{2\pi\varepsilon}\left(: e^{-i\phi_-(x)} :: e^{-i\phi_+(x)} : + : e^{i\phi_+(x)} :: e^{i\phi_-(x)} :\right) \\
&= \frac{1}{2\pi\varepsilon}\left(: e^{-i\phi_-(x)-i\phi_+(x)} : e^{-[\phi_-(x),\phi_+(x)]} + : e^{i\phi_+(x)+i\phi_-(x)} : e^{-[\phi_+(x),\phi_-(x)]}\right) \\
&= \frac{1}{2\pi\varepsilon}\left(: e^{-i\phi_-(x)-i\phi_+(x)} : e^{-\frac{i}{4\beta^2}} + : e^{i\phi_+(x)+i\phi_-(x)} : e^{\frac{i}{4\beta^2}}\right) \\
&= \frac{1}{2\pi\varepsilon}\left(: e^{-i\phi_-(x)-i\phi_+(x)} : e^{-i\pi} + : e^{i\phi_+(x)+i\phi_-(x)} : e^{i\pi}\right) \\
&= -\frac{1}{2\pi\varepsilon}\left(: e^{-i\phi(x)} : + : e^{i\phi(x)} :\right) \\
&= -\frac{1}{\pi\varepsilon} : \cos\phi :,
\end{aligned} \tag{37}
$$

where in the third line we used BCH formula, in the fourth line we used that $[\phi_+(x), \phi_-(y)] = -\frac{i}{4\beta^2}$ (this can be shown from (28) and (27)), in the fifth line we replaced $\beta^2 = \frac{1}{4\pi}$ and in the sixth line we used that $\phi(x) = \phi_+(x) + \phi_-(x)$. From now on, normal ordering will be assumed and we ommit the : : notation,

So, in this way we can build the "bosonization dictionary" that allows us to see explicitly how a compact boson theory relates to fermion theory in 2D:

$$
\begin{aligned}
\text{mass term}: &\quad \bar{\psi}\psi \quad \longleftrightarrow \quad -\frac{1}{\pi\varepsilon}\cos\phi \\
\text{chiral mass}: &\quad i\bar{\psi}\gamma^3\psi \quad \longleftrightarrow \quad -\frac{1}{\pi\varepsilon}\sin\phi \\
\text{vector current}: &\quad j_V^\mu = \bar{\psi}\gamma^\mu\psi \quad \longleftrightarrow \quad -j^\mu_{wind} = -\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi \\
\text{axial current}: &\quad j_A^\mu = \bar{\psi}\gamma^\mu\gamma^3\psi \quad \longleftrightarrow \quad -j^\mu_{shift} = -2\beta^2\partial^\mu\phi
\end{aligned} \tag{38}
$$

2 Massless Schwinger Model

Coupling a massless two-dimensional fermion to a $U(1)$ gauge field, we have QED_2 or the so called massless Schwinger Model (mSM). The action is

$$
\begin{aligned}
S &= \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{\theta}{2\pi}F_{01} + i\bar{\psi}\gamma^\mu D_\mu\psi\right) \\
&= \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{\theta}{2\pi}F_{01} + i\bar{\psi}\gamma^\mu\partial_\mu\psi + eA_\mu j_V^\mu\right).
\end{aligned} \tag{39}
$$

If the coupling is small (or heavy fermions) $m^2 \gg e^2$, then we can use perturbation theory to solve the model. However, when the theory is strongly coupled $m^2 \ll e^2$, we need to look for other methods, one of them is Bosonization. Therefore, applying the dictionary (38), the bosonized version of the mSM is:

$$
\begin{aligned}
S &= \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{\theta}{2\pi}F_{01} + \frac{1}{8\pi}(\partial_\mu\phi)^2 + \frac{e}{2\pi}A_\mu\epsilon^{\mu\nu}\partial_\nu\phi \right) \\
&= \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{(\phi+\theta)}{2\pi}F_{01} + \frac{1}{8\pi}(\partial_\mu\phi)^2 \right),
\end{aligned} \tag{40}
$$

where in the second line we integrated by parts. The corresponding classical EoM for ϕ are

$$
\begin{aligned}
\partial_\mu\partial^\mu\phi &= 2F_{01} \\
&= -2\pi\partial_\mu j_A^\mu \\
\implies \partial_\mu j_A^\mu &= -\frac{1}{\pi}F_{01} = \frac{1}{\pi}E(x),
\end{aligned} \tag{41}
$$

where we used the bosonized version of the axial current and $F^{01} = E(x)$. Note that we have an anomaly, since the axial current is not conserved anymore, thus we have a spontaneous symmetry breaking (SSB) of the chiral symmetry. Though there are no Goldstone bosons that appear as a consequence of this SSB because Goldstone theorem needs a conservation of the corresponding local current and in this case it suffers an anomaly. Similarly, the EoM for the gauge field are:

$$
\begin{aligned}
\partial_x \left(F_{01} + \frac{e}{2\pi}(\phi+\theta) \right) &= 0 \\
\implies E(x) &= \frac{e}{2\pi}(\phi+\theta),
\end{aligned} \tag{42}
$$

where we assumed vanishing field at spatial infinity. We can see again that the role of θ is as a background electric field, as before.

Coming back to our action

$$
\begin{aligned}
S &= \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{(\phi+\theta)}{2\pi}F_{01} + \frac{1}{8\pi}(\partial_\mu\phi)^2 \right) \\
&= \int d^2x \left(\frac{1}{8\pi}(\partial_\mu\phi)^2 - \frac{e^2}{8\pi^2}(\phi+\theta)^2 \right) \\
&= \int d^2x \left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{e^2}{2\pi}\phi^2 \right),
\end{aligned} \tag{43}
$$

where in the second line we integrated out the gauge fields (by completing the square) and in the third line we shifted $\phi \to \phi - \theta$ and then rescaled it $\phi \to 2\sqrt{\pi}\phi$. Since θ is irrelevant, our theory is degenerate with respect to $\theta \in [0, 2\pi]$. Finally, the theory we are left with is that of a free real scalar with mass $M_1^2 = \frac{e^2}{\pi}$; this tells us that mSM has a gap with value M_1.

Alternatively, we could have integrating out the scalar field from the action, let's do that

$$
\begin{aligned}
S &= \int d^2x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e}{2}\frac{\theta}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} + \frac{1}{8\pi}(\partial_\mu\phi)^2 - \frac{e}{2\pi}A_\nu\epsilon^{\mu\nu}\partial_\mu\phi \right) \\
&= \int d^2x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e}{2}\frac{\theta}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} - \frac{e^2}{2\pi}\epsilon^{\mu\nu}\epsilon_{\mu\alpha}A_\nu A^\alpha \right) \\
&= \int d^2x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e}{2}\frac{\theta}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} + \frac{e^2}{2\pi}A_\mu A^\mu \right),
\end{aligned} \tag{44}
$$

where in the second line we integrated out the scalar field, in the third line we used that $\epsilon^{\mu\nu} = -\epsilon_{\mu\nu}$ and that the Levi Civita tensor is fully antisymmetric. We can see that the gauge field has "acquired a mass" $M_1 = \frac{e}{\sqrt{\pi}}$, equivalent as the prescription we had before. This tells us that if we couple an external current to the theory, then there is no long range force (there is a screening of the corresponding charge see Appendix A).

3 Massive Schwinger Model (MSM)

If we turn on the mass of the fermions, the bosonized action gets modified by a periodic potential

$$\begin{aligned} S &= \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{(\phi+\theta)}{2\pi}F_{01} + \frac{1}{8\pi}(\partial_\mu\phi)^2 + \frac{m}{\pi\varepsilon}\cos\phi \right) \\ &= \int d^2x \left(\frac{1}{8\pi}(\partial_\mu\phi)^2 - \frac{e^2}{8\pi^2}(\phi+\theta)^2 + \frac{m}{\pi\varepsilon}\cos\phi \right) \\ &= \int d^2x \left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{e^2}{2\pi}\phi^2 + mce\cos\left(2\sqrt{\pi}\phi - \theta\right) \right), \end{aligned} \tag{45}$$

where in the second line we integrated out the gauge field, in the third line we shifted $\phi \to \phi - \theta$ and then rescaled it $\phi \to 2\sqrt{\pi}\phi$. And $ce = \frac{1}{\pi\varepsilon}$ (note that the mass that appears in the action is the bare mass). This theory is now not exactly solvable and breaks explicitly chiral symmetry.

When we are in the strong coupling limit, i.e. $e \gg m$, then the self couplings in the cosine potential are weak and we can treat them perturbatively. Thus, the theory alwas contains at least one particle: a meson with mass $M_1 = \frac{e}{\sqrt{\pi}}$. And in this regime of small mass, Caroll, Kogut, Sinclair, and Susskind '75, found that the lowest mass state (vector state[2]) is

$$\frac{M_1}{e} = \frac{1}{\sqrt{\pi}} + e^{\gamma}\frac{m}{e} + \dots \tag{46}$$

While the next lower mass state, corresponding to a bound scalar state

$$\frac{M_2}{M_1} = 2 - \#\left(\frac{m}{e}\right)^2 + \dots \tag{47}$$

On the other hand, when we consider the weak coupling limit $e \ll m$, then it is easier to work in the Fermi form (not use the bosonization dictionary), here we have a theory of quarks with weak Coulomb interaction:

$$S = \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{\theta}{2\pi}F_{01} + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + eA_\mu\bar{\psi}\gamma^\mu\psi \right). \tag{48}$$

And here we can treat the theory as non-relativistic...

4 Lattice Hamiltonian Approach to Schwinger Model

In the aim to solve this model efficiently, we put the system on a lattice. To do this, define the angular variables $\theta_\mu(n)$ such that it lives in a link that joins n with $n + \mu$ in the $\hat{\mu}$ direction:

[2] We will see in the next section why this is related to a vector state.

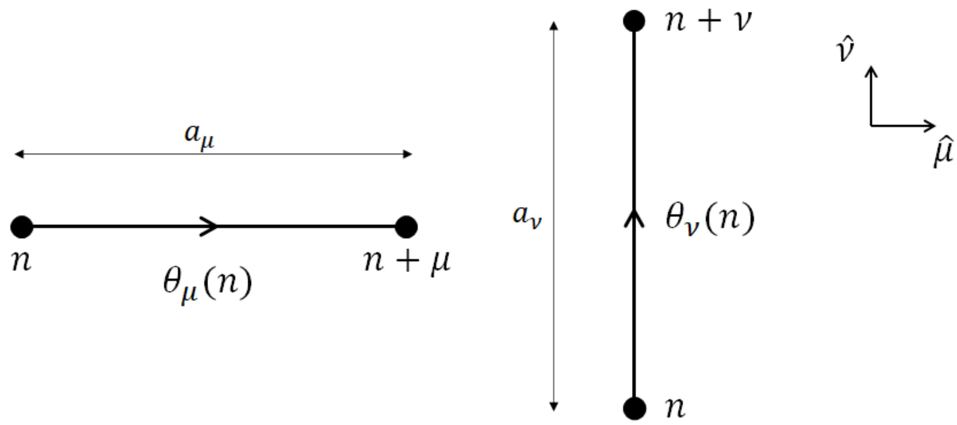

Figure 2: Angular variables live on links.

And $\theta_{-\mu}(n+\mu) = -\theta_\mu(n)$. We want to define an action that is invariant under gauge transformations that are defined as follows:

$$
\begin{aligned}
\theta_\mu(n) &\to \theta_\mu(n) - \chi(n) \\
\theta_\mu(n) = -\theta_{-\mu}(n+\mu) &\to -\theta_{-\mu}(n+\mu) + \chi(n+\mu) = \theta_\mu(n) + \chi(n+\mu),
\end{aligned} \tag{49}
$$

where $\chi(n)$ is an arbitrary function at the site n. Therefore, the net effect of two separate gauge transformations on n and $n+\mu$:

$$
\theta_\mu(n) \to \theta_\mu(n) + \chi(n+\mu) - \chi(n) = \theta_\mu(n) + \Delta_\mu \chi(n). \tag{50}
$$

Now we need to construct gauge invariant quantities, one such an object is

$$
\begin{aligned}
\theta_{\mu\nu}(n) &= \Delta_\mu \theta_\nu(n) - \Delta_\nu \theta_\mu(n) \\
&= \theta_\nu(n+\mu) - \theta_\nu(n) - \theta_\mu(n+\nu) + \theta_\mu(n) \\
&= \theta_\mu(n) + \theta_\nu(n+\mu) + \theta_{-\mu}(n+\nu+\mu) + \theta_{-\nu}(n+\nu) \\
&\to \theta_\nu(n+\mu) + \chi(n+\nu+\mu) - \chi(n+\mu) - \theta_\nu(n) - \chi(n+\nu) + \chi(n) - \\
&- \theta_\mu(n+\nu) - \chi(n+\nu+\mu) + \chi(n+\nu) + \theta_\mu(n) + \chi(n+\mu) - \chi(n) \\
&= \theta_{\mu\nu}(n).
\end{aligned} \tag{51}
$$

This is a plaquette:

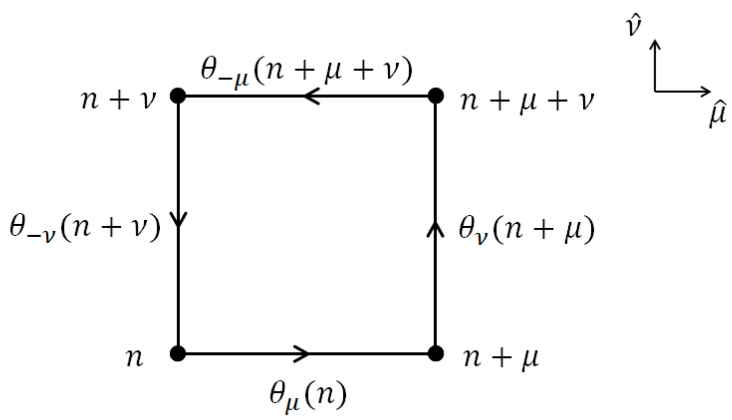

Figure 3: A gauge invariant quantity is a plaquette.

We can make the analogy of this quantity with the electromagnetic strength tensor $F_{\mu\nu}$. Now, we define the gauge invariant action

$$
\begin{aligned}
S &= J \sum_{n,\mu,\nu} \left[1 - \cos\theta_{\mu\nu}(n)\right], \quad \theta_{\mu\nu}(n) \ll 1,\ a \to 0 \\
&\approx J \int \frac{d^2x}{a^2} \frac{1}{2} \theta_{\mu\nu}^2, \qquad (52)
\end{aligned}
$$

where a is the spatial lattice spacing. If we take $\theta_{\mu\nu} = aeF_{\mu\nu}$ and $J = \frac{1}{2e^2}$, we recover the EM action.

Since we are interested in finding the spectrum of the Schwinger model, it is better to work directly with the Hamiltonian. To find it, let us choose $\mu = \tau$, $\nu = x$ and the gauge where $\theta_\tau(n_x, n_\tau) = 0$ (this will correspond to choose $A_0 = 0$ in the continuum case). Then, $\theta_{\tau x}(n_x, n_\tau) = \theta_x(n_x, n_\tau + \tau) - \theta_x(n_x, n_\tau)$. We have to be aware that we have two lattice spacings, in general, one for space (a) and one for time (a_τ), so in the limit of $a_\tau \to 0$:

$$
\begin{aligned}
\theta_{\tau x}(n_x, n_\tau) &\approx \dot{\theta}_x(n_x, n_\tau) a_\tau \\
\implies 1 - \cos\theta_{\tau x}(n_x, n_\tau) &\approx \frac{1}{2} \dot{\theta}_{\tau x}^2(n_x, n_\tau) a_\tau^2
\end{aligned}
$$

Leaving the action as:

$$
S \approx a_\tau \sum_{n_\tau} \frac{a_\tau J}{2} \sum_{n_x} \dot{\theta}_{\tau x}^2(n_x, n_\tau). \qquad (53)
$$

Consider the amplitude,

$$
\begin{aligned}
Z &= \int \mathcal{D}\theta_x \; e^{-S}, \quad \mathcal{D}\theta_x = \prod_{n_\tau} \prod_{n_x} d\theta_x(n_x, n_\tau) \\
&= \int \left(\prod_{n_x} d\theta_x(n_x, n_\tau = 1) \right) \dots \left(\prod_{n_x} d\theta_x(n_x, n_\tau = N_\tau) \right) \; e^{-S}, \quad d\theta_x^{n_\tau} = \left(\prod_{n_x} d\theta_x(n_x, n_\tau) \right) \\
&= \int d\theta_x^1 \dots d\theta_x^{N_\tau} \; e^{-S}, \qquad (54)
\end{aligned}
$$

consider the matrix element

$$
\begin{aligned}
\langle \theta_x^{n_\tau+1} | \, \hat{T} \, | \theta_x^{n_\tau} \rangle = T(\theta_x^{n_\tau+1}, \theta_x^{n_\tau}) &= \exp \left[-\frac{J a_\tau}{2a_\tau^2} \sum_{n_x} \left(\theta_x(n_x, n_\tau + 1) - \theta_x(n_x, n_\tau)\right)^2 \right], \\
\implies Z &= \int \prod_{n_\tau} \left[d\theta_x^{n_\tau} T(\theta_x^{n_\tau+1}, \theta_x^{n_\tau}) \right].
\end{aligned}
$$

Now, given our angular variable $\theta_x(n_x)$, let the corresponding canonical momentum operator be $\hat{L}(n_x)$, namely it obeys the commutation relation

$$
[\hat{\theta}_x(n_x), \hat{L}(n'_x)] = i\delta_{n_x, n'_x}. \qquad (55)
$$

Then, consider the transition element:

$$
\begin{aligned}
\langle\theta'_x(n'_x)|\sum_{n_x}\hat{L}^2(n_x)\,|\theta_x(n_x)\rangle &= \int dLdL'\prod_{n_x}\langle\theta'_x(n_x)|L'(n_x)\rangle\,\langle L'(n_x)|e^{\hat{L}(n_x)^2}|L(n_x)\rangle\,\langle L(n_x)|\theta_x(n_x)\rangle \\
&= \int dL\prod_{n_x} e^{iL(n_x)[\theta'_x(n_x)-\theta_x(n_x)]+L(n_x)^2} \\
&= \mathcal{N}\prod_{n_x} e^{\frac{1}{4}[\theta'_x(n_x)-\theta_x(n_x)]^2}, \quad \mathcal{N} = \text{result of Gaussian integral over } L,
\end{aligned}
$$

where we have used that

$$
\langle\theta_x(n_x)|L(n_x)\rangle = e^{iL(n_x)\theta_x(n_x)}. \tag{56}
$$

With this, we can identify the operator $\hat{T}$ with $\exp\left(\sum_{n_x}\hat{L}^2(n_x)\right)$ as follows

$$
\hat{T} = \exp\left[-\frac{2J}{a_\tau}\sum_{n_x}\hat{L}^2(n_x)\right]. \tag{57}
$$

Now, clearly operator $e^{-a_\tau\hat{H}} = \hat{T}$ represents the evolution in one discrete time of the variables θ_x, thus

$$
\begin{aligned}
\hat{H} &= -\frac{1}{a_\tau}\log\hat{T} = \frac{2J}{a_\tau^2}\sum_{n_x}\hat{L}^2(n_x) \\
&= \frac{\beta}{2}\sum_{n_x}\hat{L}^2(n_x), \quad \beta\equiv\frac{4J}{a_\tau^2}.
\end{aligned} \tag{58}
$$

From now on, we will only focus on the spatial spacings $n_x = n$ and we can omit the time ones. Let's study more closely to the $\theta_x(n_x) = \theta(n)$ and $\hat{L}(n)$. We know that $\theta(n)$ is a periodic variable and as such, its canonical momentum is an angular momentum operator, so $\hat{L}(m)$ are generators of $\theta(m)$ (generates angular translations), then

$$
\begin{aligned}
e^{i\theta(m)L(m)} &= e^{i(\theta(m)+2\pi)L(m)} \\
\implies e^{i2\pi L(m)} &= 1 \\
L(m)\,|l(m)\rangle &= l(m)\,|l(m)\rangle\,, \quad l(m) = 0,\pm1,\pm2,\ldots\ L \text{ has discrete spectrum.}
\end{aligned} \tag{59}
$$

Since $[L(m),\theta(m)] = -i$, then one can show by induction that $[L(m),\theta(m)^n] = -in\theta(m)^{n-1}$, therefore

$$
\begin{aligned}
L(m)e^{\pm i\theta(m)}\,|l(m)\rangle &= \sum_k L(m)\frac{(\pm i\theta(m))^k}{k!}\,|l(m)\rangle \\
&= -i\sum_k\frac{(\pm i)^k\theta(m)^{k-1}k}{k!}\,|l(m)\rangle + e^{\pm i\theta(m)}L(m)\,|l(m)\rangle \\
&= \pm\sum_k\frac{(\pm i\theta(m))^{k-1}}{(k-1)!}\,|l(m)\rangle + e^{\pm i\theta(m)}l(m)\,|l(m)\rangle \\
&= (l\pm1)\,|l(m)\rangle\,,
\end{aligned} \tag{60}
$$

so $e^{\pm i\theta(m)}|l(m)\rangle = |l \pm 1\rangle$ corresponding to raising and lowering operators, respectively. If we make the analogy with electromagnetism, in the $A_0 = 0$ gauge, we know that $[E(x), A_1(x')] = i\delta(x - x')$, whose lattice version is $[E(m), A_1(m')] = \frac{i}{a}\delta_{m,m'}$. Thus by letting $\theta(m) = aeA_1(m)$, then

$$E(m) = eL(m), \quad L \text{ corresponds to the electric field on links.} \tag{61}$$

Coming back to the Hamiltonian, to fix β, we take the continuum limit and it should give us the energy for a constant electric field

$$\begin{aligned} &H \xrightarrow[a\to 0]{} \frac{1}{2}\int dx E(x)^2 \\ &\Longrightarrow \beta = ae^2. \end{aligned} \tag{62}$$

Leaving us the EM Hamiltonian, introducing a background field,

$$H = \frac{a}{2}\sum_n [E(n) + F]^2 \tag{63}$$

Let us turn our attention to the fermions, for simplicity, we are going to use the Dirac representation (in contrast to the Weyl one used in the previous sections), viz.

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{64}$$

And the Dirac Hamiltonian is

$$\begin{aligned} H &= \Pi_\psi \dot{\psi} - \mathcal{L}, \qquad\qquad \Pi_\psi = i\psi^\dagger \\ &= -i\bar{\psi}\gamma^1 \frac{\partial}{\partial x}\psi + m\bar{\psi}\psi, \end{aligned} \tag{65}$$

where each field has the 2-component spinor form

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix},$$

hence

$$H = -i\psi_d^\dagger \partial_x \psi_u - i\psi_u^\dagger \partial_x \psi_d + m\psi_u^\dagger \psi_u - m\psi_d^\dagger \psi_d. \tag{66}$$

In order to eliminate the fermion doubling problem (Appendix C) in 2D, we employ the 'staggered' formulation, where

$$\begin{aligned} \psi_u &\to \phi(n), \quad \text{if } n \text{ is even} \\ \psi_d &\to \phi(n), \quad \text{if } n \text{ is odd} \end{aligned}$$

together with the anti-commutation relations

$$\left\{\phi^\dagger(n),\phi(m)\right\}=\delta_{nm},\qquad \{\phi(n),\phi(m)\}=0. \tag{67}$$

We can define the spatial derivative as

$$\frac{\partial\psi_{u/d}(n)}{\partial x}=-\left[\frac{\phi(n+1)-\phi(n-1)}{2a}\right], \tag{68}$$

so that Dirac equation writes:

$$\begin{aligned}
(i\gamma^\mu\partial_\mu-m)\psi=i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\dot\psi_u\\\dot\psi_d\end{pmatrix}+i\begin{pmatrix}0&1\\-1&0\end{pmatrix}\partial_x\begin{pmatrix}\psi_u\\\psi_d\end{pmatrix}-m\begin{pmatrix}\psi_u\\\psi_d\end{pmatrix}=0\\
0=i\begin{pmatrix}\dot\psi_u\\\dot\psi_d\end{pmatrix}+i\begin{pmatrix}0&1\\1&0\end{pmatrix}\partial_x\begin{pmatrix}\psi_u\\\psi_d\end{pmatrix}-m\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\psi_u\\\psi_d\end{pmatrix}\\
0=i\dot\psi_u+i\partial_x\psi_d-m\psi_u\\
0=i\dot\psi_d+i\partial_x\psi_u+m\psi_d,
\end{aligned} \tag{69}$$

or in its lattice form:

$$\begin{aligned}
i\dot\phi(n)-m\phi(n)=i\left[\frac{\phi(n+1)-\phi(n-1)}{2a}\right],\quad n-\text{even}\\
i\dot\phi(n)+m\phi(n)=i\left[\frac{\phi(n+1)-\phi(n-1)}{2a}\right],\quad n-\text{odd}
\end{aligned} \tag{70}$$

Therefore, the Dirac Hamiltonian reads:

$$\begin{aligned}
H=&\sum_{n-odd}\left[\frac{i\phi^\dagger(n)}{2a}(\phi(n+1)-\phi(n-1))+\frac{i\phi^\dagger(n+1)}{2a}(\phi(n+2)-\phi(n))+\right.\\
&\left.+m\phi^\dagger(n+1)\phi(n+1)-m\phi^\dagger(n)\phi(n)\right]\\
=&\sum_{n-odd}\left[\frac{i}{2a}(\phi^\dagger(n)\phi(n+1)+\phi^\dagger(n+1)\phi(n+2)-\phi^\dagger(n)\phi(n-1)-\phi^\dagger(n+1)\phi(n))+\right.\\
&\left.+m\phi^\dagger(n+1)\phi(n+1)-m\phi^\dagger(n)\phi(n)\right]\\
=&\sum_{n-odd}\left[\frac{i}{2a}\left(\phi^\dagger(n)\phi(n+1)-\phi^\dagger(n+1)\phi(n)\right)-m\phi^\dagger(n)\phi(n)\right]+\\
&\sum_{n-even}\left[\frac{i}{2a}\left(\phi^\dagger(n)\phi(n+1)-\phi^\dagger(n+1)\phi(n)\right)+m\phi^\dagger(n)\phi(n)\right]\\
=&\sum_{n}\left[\frac{i}{2a}\left(\phi^\dagger(n)\phi(n+1)-\phi^\dagger(n+1)\phi(n)\right)+(-1)^n m\phi^\dagger(n)\phi(n)\right].
\end{aligned} \tag{71}$$

What about gauge invariance? Naively, we if we consider

$$\begin{aligned}
\phi(n)\to e^{-i\chi(n)}\phi(n)\\
A_\mu(n)\to A_\mu(n)+\frac{1}{ea}\Delta_\mu\chi(n),
\end{aligned}$$

then

$$\phi^\dagger(n)\phi(n) \to \phi^\dagger(n)e^{i\chi(n)}e^{-i\chi(n)}\phi(n) = \phi^\dagger(n)\phi(n), \quad \text{but}$$
$$\phi^\dagger(n)\phi(n+1) \to \phi^\dagger(n)e^{i\chi(n)}e^{-i\chi(n+1)}\phi(n) \neq \phi^\dagger(n)\phi(n+1)$$

Then, in order for the second term to be gauge invariant, we have to be careful and follow Schwinger. Schwinger tells us that if instead we use the operator (recall we are in $A_0(n) = 0$ gauge):

$$\phi^\dagger e^{ieaA_1(n)}\phi(n+1) \to \phi^\dagger e^{i\chi(n)}e^{ieaA_1(n)+i\chi(n+1)-i\chi(n)}e^{-i\chi(n+1)}\phi(n+1) = \phi^\dagger e^{ieaA_1(n)}\phi(n+1),$$

we recover gauge invariance. Thus, the full MSM Hamiltonian is then[3]

$$\begin{aligned} H = \frac{i}{2a}\sum_n \left[\phi^\dagger(n)e^{i\theta(n)}\phi(n+1) - \phi^\dagger(n+1)e^{-i\theta(n)}\phi(n)\right] + \\ + m_{latt}\sum_n (-1)^n\phi^\dagger(n)\phi(n) + \frac{e^2a}{2}\sum_n\left[L(n)+\frac{F}{e}\right]^2, \quad F = \frac{e\theta}{2\pi},\ m_{latt} = m - \frac{e^2a}{8}. \end{aligned} \tag{72}$$

Numerically, it is more convenient to write the QED$_2$ Hamiltonian in a spin form. This can be done by implementing the Jordan-Wigner transformation:

$$\begin{aligned} \phi(n) &= \prod_{\ell<n}[i\sigma_3(\ell)]\sigma^-(n) \\ \phi^\dagger(n) &= \prod_{\ell<n}[-i\sigma_3(\ell)]\sigma^+(n), \end{aligned} \tag{73}$$

then the terms

$$\begin{aligned} i)\ \phi^\dagger(n)e^{i\theta(n)}\phi(n+1) &= \prod_{\ell<n}[-i\sigma_3(\ell)]\sigma^+(n)e^{i\theta(n)}\prod_{\ell'<n+1}[i\sigma_3(\ell)]\sigma^-(n+1) \\ &= i\sigma^+(n)e^{i\theta(n)}\sigma_3(n)\sigma^-(n+1) \\ &= i\sigma^+(n)\sigma_3(n)e^{i\theta(n)}\sigma^-(n+1) \\ &= -i\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1) \end{aligned} \tag{74}$$

$$\begin{aligned} ii)\ \phi^\dagger(n+1)e^{-i\theta(n)}\phi(n) &= \prod_{\ell<n+1}[-i\sigma_3(\ell)]\sigma^+(n+1)e^{-i\theta(n)}\prod_{\ell'<n}[i\sigma_3(\ell)]\sigma^-(n) \\ &= -i\sigma^+(n+1)\sigma_3(n+1)e^{-i\theta(n)}\sigma^-(n) \\ &= -i\sigma^+(n+1)e^{-i\theta(n)}\sigma_3(n+1)\sigma^-(n) \\ &= i\sigma^+(n+1)e^{-i\theta(n)}\sigma^-(n) \end{aligned} \tag{75}$$

[3]Notice that the fermion mass m gets modified on a lattice by m_{latt}, where $m_{latt} = m - \frac{e^2a}{8}$ (Dempsey, Pufu, Klebanov, Zan '22).

$$\begin{aligned} iii)\ \phi^\dagger(n)\phi(n) &= \prod_{\ell<n}[-i\sigma_3(\ell)]\sigma^+(n)\prod_{\ell'<n}[i\sigma_3(\ell)]\sigma^-(n) \\ &= \sigma^+(n)\sigma^-(n) \\ &= \frac{1}{2}\left[1+\sigma_3(n)\right]. \end{aligned} \tag{76}$$

Therefore,

$$H_{QED_2} = \frac{1}{2a}\sum_n\left[\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1)+\sigma^+(n+1)e^{-i\theta(n)}\sigma^-(n)\right]+ \\ +\frac{m_{latt}}{2}\sum_n(-1)^n\left[1+\sigma_3(n)\right]+\frac{e^2a}{2}\sum_n\left[L(n)+\frac{F}{e}\right]^2, \tag{77}$$

or equivalently, the dimensionless Hamiltonian

$$W=\frac{2}{ae^2}H_{QED_2}=\sum_n\left[L(n)+\frac{F}{e}\right]^2+\frac{\mu}{2}\sum_n(-1)^n\sigma_3(n)+ \\ +x\sum_n\left[\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1)+\text{c.c.}\right],\quad \mu=\frac{2m_{latt}}{ae^2},\quad x=\frac{1}{e^2a^2}, \tag{78}$$

where we used, assuming N is even, $\sum_n(-1)^n=0$. One can see the term accompanying x as a hopping term. Therefore, in the $x\to\infty$ limit (weak coupling), such hopping term dominates and the Hamiltonian is reminiscent of a XY chain. Thus, one expects that the ground energy per site has the value

$$\frac{\omega_0}{xN}\underset{x\to\infty}{\longrightarrow}-\frac{2}{\pi}.$$

To see what happens at strong coupling ($x\ll 1$), for simplicity, let's work with zero background field $F=0$. Here, one can choose the ground state as the one that minimizes the term $\sum_n\left[L(n)^2+\frac{\mu}{2}(-1)^n\sigma_3(n)\right]$, namely, the one that has zero electric field on every link $L(n)=0$ and spin configuration so that each spin contributes $-\mu/2$. This is

$$\frac{\mu}{2}\sum_n(-1)^n\sigma_3(n)=-\frac{N\mu}{2}, \tag{79}$$

which happens when (see Figure 4)

$$\sigma_3(n)=\begin{cases}1 & ,\quad n-\text{odd}\\ -1 & ,\quad n-\text{even}\end{cases} \tag{80}$$

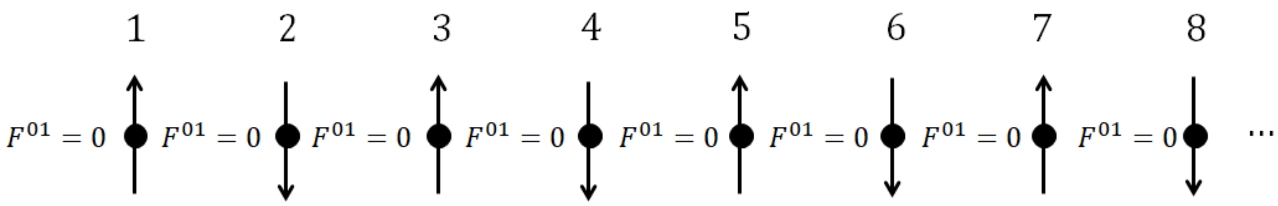

Figure 4: Ground state configuration of strongly coupled Schwinger model in its spin representation.

Or in terms of fermions this state is:

$$\phi^\dagger(n)\phi(n) = \frac{1}{2}(1+\sigma_3(n)) = \frac{1}{2}\left[1-(-1)^n\right] = \begin{cases} 1 & , \quad n - \text{odd} \\ 0 & , \quad n - \text{even} \end{cases} \tag{81}$$

that is, even sites are completely empty and odd ones are fully filled contributing with a negative energy $-\mu/2$. This is analogue to a "Dirac sea"[4]. So, we can only put fermions on even sites.

Now, it's time to turn on excitations. Before getting into its details, let's see what happens with Gauss' Law:

$$\begin{aligned} \frac{\partial E}{\partial x} = \psi^\dagger \psi \implies & E(x) - E(x_0) = Q_{Total}. \\ \implies & L(n+1) - L(n) = \phi^\dagger(n)\phi(n) - Q_{vacuum} \\ & L(n+1) - L(n) = \phi^\dagger(n)\phi(n) - \frac{1}{2}\left[1-(-1)^n\right] \\ & L(n+1) - L(n) = \frac{1}{2}\left[\sigma_3(n) + (-1)^n\right]. \end{aligned} \tag{82}$$

Notice that this law can be written as (assuming a constant incoming and outgoing field F/e):

$$L(n) = \sum_{k=1}^{n} Q_k + \frac{F}{e}, \qquad Q_k = \frac{1}{2}\left[\sigma_3(k) + (-1)^k\right], \qquad \sum_{k=1}^{N} Q_k = 0. \tag{83}$$

This means that, considering Open Boundary Conditions, we can completely eliminate the gauge dependence by fixing an initial electric field $E(0)$ and imposing that the incoming field is equal to the outgoing one (which is a consequence of charge conservation), we have $E(0)/g = L(0) + F/e = F/e = L(N) + F/e = E(N)/g$, or $L(0) = L(N) = 0$. However, this comes with the price of having a non-local, long-range term in our final Hamiltonian, viz,

$$\begin{aligned} W = \sum_{n=1}^{N}\left[\sum_{k=1}^{n}\frac{1}{2}\left(\sigma_3(k) + (-1)^k\right) + \frac{F}{e}\right]^2 + \frac{\mu}{2}\sum_{n=1}^{N}(-1)^n\sigma_3(n) + \\ + x\sum_{n=1}^{N-1}\left[\sigma^+(n)\sigma^-(n+1) + \text{c.c.}\right], \qquad \sum_{k=1}^{N} Q_k = 0. \end{aligned} \tag{84}$$

From Gauss' law, we can see that excitations on even sites will give a $Q_{total} = +1$ (antiquark excitation that corresponds to adding a fermion) and excitations on odd sites will give a $Q_{total} = -1$ (quark excitation that corresponds to not adding a fermion (Pauli exclusion principle)). With this in mind, let's study two possible excitations: 1 spin flip and 2 spin flips.

If we flip one spin, then, recalling Section 1, this induces a new electric field everywhere to the right of the flipped spin (see Figure 5). Such a configuration will cost an infinite amount of energy (Section 1), so this is not good.

If we flip 2 spins, then inserting a quark-antiquark pair (with charges $\mp e$, respectively) gives an electric field e between the two charges (flipped spins as shown in Figure 5) and zero

[4] A Dirac sea is a system whose vacuum is a state where all negative energy states are filled and none on the positive energy ones. If we want to introduce an electron, it has to be in one of the positive energy states because of Pauli exclusion principle.

everywhere else. Hence, we conclude that the lowest excitation is a quark-antiquark pair with one lattice separation (minimizes energy).

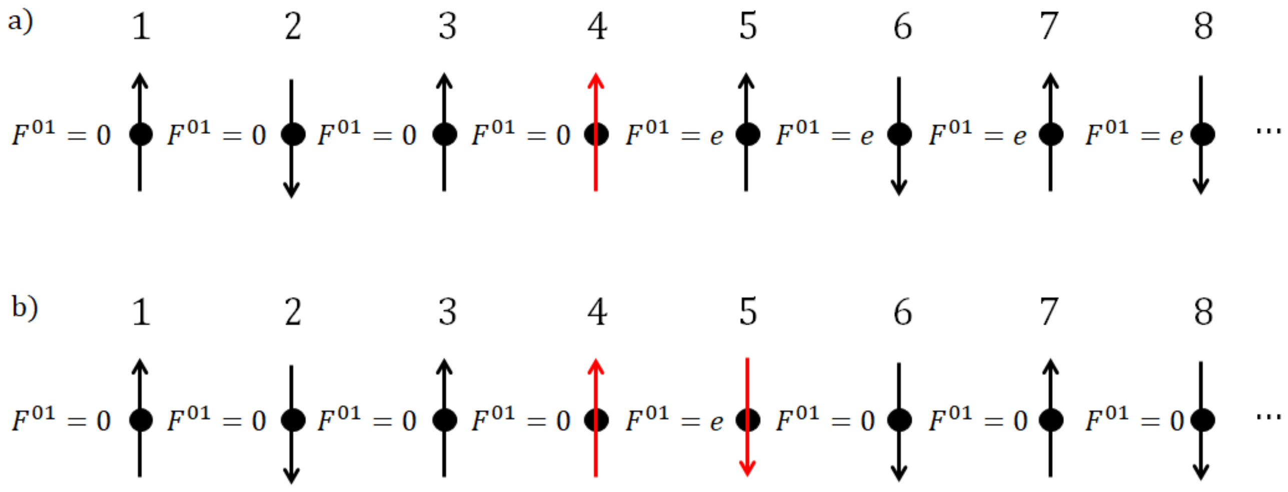

Figure 5: Excited state configurations: a) Introducing one charged particle costs an infinite amount of energy, while b) introducing a $q\bar{q}$-pair corresponds to the mimimal stable configuration.

Therefore, the excited state has the term $\phi^\dagger(n)\phi(n+1)\,|0\rangle$. And by translational symmetry, there are two choices for the excited state:

$$
\begin{aligned}
|v\rangle &= \frac{1}{\sqrt{N}}\sum_n \left[\phi^\dagger(n)\phi(n+1) + \phi^\dagger(n+1)\phi(n)\right]|0\rangle\,, \quad \text{or} \\
|s\rangle &= \frac{1}{\sqrt{N}}\sum_n \left[\phi^\dagger(n)\phi(n+1) - \phi^\dagger(n+1)\phi(n)\right]|0\rangle\,,
\end{aligned} \tag{85}
$$

where the v/s stands for vector/scalar states, correspondingly. This labeling is not random, it comes from the fact that if we calculate the axial vector and pseudo-scalar densities, we obtain:

$$
\begin{aligned}
j_5^0 = \psi^\dagger\gamma_5\psi = \psi_u^\dagger\psi_d + \psi_d^\dagger\psi_u &\longleftrightarrow \phi^\dagger(n)\phi(n+1) + \phi^\dagger(n+1)\phi(n), \quad \text{axial-vector density.} \\
\bar{\psi}\gamma_5\psi = \psi^\dagger\gamma^1\psi = \psi_u^\dagger\psi_d - \psi_d^\dagger\psi_u &\longleftrightarrow \phi^\dagger(n)\phi(n+1) - \phi^\dagger(n+1)\phi(n), \quad \text{pseudo-scalar density.}
\end{aligned} \tag{86}
$$

where

$$
\gamma_5 = \gamma^0\gamma^1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

For non-zero but finite $x \ll 1$, the vector state is the first excited state while the scalar one is the second. The energy gap between the vacuum and the vector state is the 'vector mass' M_1 and the one between the vacuum and the scalar state is the 'scalar mass', which are correspondingly:

$$
\begin{aligned}
\varepsilon_1 - \varepsilon_0 &= \frac{2}{ae^2}M_1 \\
\varepsilon_2 - \varepsilon_0 &= \frac{2}{ae^2}M_2.
\end{aligned} \tag{87}
$$

A Charge Screening in mSM

This and the following sections are based on Coleman, Hackiw, Susskind '75.

Let's couple our massless fermion to an external (conserved) current with zero total charge (two opposite charges of magnitude Q separated by a distance L), define this current to be $J^\mu = \epsilon^{\mu\nu}\partial_\nu D$, with D a scalar complex number field that describes a static distribution separated a distance L. So, $D = Q$ in the region $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$. So, the Lagrangian becomes

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + eA_\mu\bar{\psi}\gamma^\mu - A_\mu J^\mu. \tag{88}$$

Then, the extra vacuum contributions, at leading order, coming from the current (as shown in Figure 6) in the bosonized version are:

$$\begin{aligned}
S_a &= -\frac{1}{2}\int d^2x J^\mu \Box^{-2} J_\mu \\
&= -\frac{1}{2}\int d^2x \epsilon^{\mu\nu}\epsilon_{\mu\alpha}(\partial_\nu D)\Box^{-2}(\partial^\alpha D) \\
&= \frac{1}{2}\int d^2x (\partial_\mu D)\Box^{-2}(\partial^\mu D) \\
&= -\frac{1}{2}\int d^2x\ D\partial_\mu \Box^{-2}\partial^\mu D \\
&= -\frac{1}{2}\int d^2x\ D^2 \quad \text{constant of the action} \\
S_b &= -e\int d^2x J^\mu \Box^{-2} j^V_\mu \\
&= -\frac{e}{2\pi}2\sqrt{\pi}\int d^2x \epsilon^{\mu\nu}\epsilon_{\mu\alpha}(\partial_\nu D)\Box^{-2}(\partial^\alpha \phi) \\
&= \frac{e}{\sqrt{\pi}}\int d^2x (\partial_\mu D)\Box^{-2}(\partial^\mu \phi) \\
&= -\frac{1}{2}\int d^2x\ D\partial_\mu \Box^{-2}\partial^\mu \phi \\
&= -\frac{1}{2}\int d^2x\ D\phi \quad \text{shifts the field in the mass term,}
\end{aligned} \tag{89}$$

where the $2\sqrt{\pi}$ term in the second line for S_b comes from the fact that we rescaled our scalar field $\phi \to 2\sqrt{\pi}\phi$.

So, the original and the new Hamiltonians are:

$$\begin{aligned}
\mathcal{H}_o &= \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + \frac{e^2}{\pi}\phi^2\right] \implies \mathcal{E}(0) = \text{constant vacuum energy/unit length} \\
\mathcal{H}_{new} &= \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + \frac{e^2}{\pi}\left(\phi + \frac{2\sqrt{\pi}}{e}D\right)\phi + D^2\right], \quad \phi' = \phi + \frac{\sqrt{\pi}}{e}D \\
&= \frac{1}{2}\left[\pi'^2 + (\nabla\phi')^2 + \frac{e^2}{\pi}\phi'^2\right] \implies \mathcal{E}'(0) = \mathcal{E}(0) \text{ (since vacuum is degenerate),}
\end{aligned} \tag{90}$$

thus the total vacuum interaction energy for large L is $E = (\mathcal{E}'(0) - \mathcal{E}(0))L = 0$. That is, long range force is zero and the charge is shielded (which an explicit form of what we argued by showing that the photon "gets" a mass).

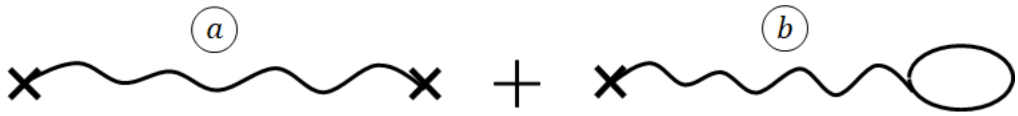

Figure 6: Leading order vacuum corrections added due to the presence of an external current.

B Charge Screening in MSM

Let's follow the same procedure as the previous section, so couple our fermions to an external (conserved) current with zero total charge (two opposite charges of magnitude Q separated by a distance L), define this current to be $J^\mu = \epsilon^{\mu\nu}\partial_\nu D$, with D a scalar complex number field that describes a static distribution separated a distance L. So, $D = Q$ in the region $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$. So, the Lagrangian becomes

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - -m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + eA_\mu\bar{\psi}\gamma^\mu - A_\mu J^\mu. \tag{91}$$

Then, the extra vacuum contributions, at leading order, coming from the current (as shown in Figure 6) in the bosonized version are the same as before.

So, the new Hamiltonian is:

$$\begin{aligned}
\mathcal{H}_{new} &= \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + \frac{e^2}{\pi}\left(\phi + \frac{2\sqrt{\pi}}{e}D\right)\phi + D^2\right] - mce\cos\left(2\sqrt{\pi}\phi - \theta\right), \quad \phi' = \phi + \frac{\sqrt{\pi}}{e}D \\
&= \frac{1}{2}\left[\pi'^2 + (\nabla\phi')^2 + \frac{e^2}{\pi}\phi'^2\right] - mce\cos\left(2\sqrt{\pi}\phi' - \theta - 2\pi Q/e\right) \\
\Longrightarrow \mathcal{E}'(\theta) &= \mathcal{E}\left(\theta + \frac{2\pi Q}{e}\right) \text{ (vacuum energy density),}
\end{aligned} \tag{92}$$

with $\mathcal{E}(\theta)$ being the vacuum energy density in the absence of external charges. Thus, the total vacuum interaction energy for large L is

$$E = (\mathcal{E}'(\theta) - \mathcal{E}(\theta))L = \begin{cases} = 0 & , \quad Q = ne, \ n \in \mathbf{Z} \\ \neq 0 & , \quad Q = ne, \ n \notin \mathbf{Z} \end{cases}. \tag{93}$$

That is, long range force is zero and the charge is shielded when the external charges are multiples of the fundamental charge e. Otherwise, long-range force is present (non-zero string tension occurs) and the individual charges are confined.

C Fermion doubling problem