SYK Comments

March 11, 2022

This series of notes are compiled from 3 main papers: Maldacena and Stanford (Comments on SYK), Polchinsky and Rosenhaus (The Spectrum of SYK model) and Sarosi (AdS_2, Holography and the SYK model).

1 Characteristics and importance

The SYK model describes a 1-D system of N (Majorana) fermions with an all-to-all random quartic interaction. It has the following important features:

1. It is solvable at strong coupling at large N.
2. It is maximally chaotic. The Lyapunov exponent λ is defined by an Out of Time Order (OTO) 4-point function; in Einstein gravity $\lambda = \frac{2\pi}{\beta}$ for a Black Hole (BH) and this is the maximal allowed value for λ. SYK saturates this bound.
3. At low energies, an emergent conformal symmetry manifests in the 2-point functions.

So, why is SYK important to study? First of all, due to the scarcity of non-trivial systems that can be solved at strong coupling, makes it worth to consider. Besides this, in the context of classical chaos 1. and 2. do not imply that the model is solvable (integrable); however in quantum systems, there is no restriction, as in this case. Finally, 2. and 3. suggests a holographic dual that could be Einstein gravity in some form.

In summary, these 3 important aspects potentially classify SYK as a solvable model of holography.

2 Two-point function

Let us start by writing the SYK model:

$$\mathcal{L}_{SYK} = -\frac{1}{2}\sum_{i=1}^{N} \chi_i \partial_\tau \chi_i - \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l, \tag{1}$$

where $\{\chi_i \chi_j\} = \delta_{ij}$ and has quenched disorder [1] with random couplings J_{ijkl} taken from a Gaussian distribution:

[1]Quenched Disorder (QD) describes a statistical mechanical system with some parameters that are random variables that do not evolve in time (quenched, frozen), e.g. Spin Glasses. It is hard to analyze and the common method of approach for it is the replica trick. In contrast, Annealed Disorder (AD) describes a system whose random parameters evolve in time (this evolution is related to that of the dof of the system). This is easier to analyze than QD.

$$P(J_{ijkl}) \sim \exp\left(-\frac{N^3 J_{ijkl}^2}{2(3! J^2)}\right),$$

that is $\overline{J_{ijkl}} = 0$ and $\overline{J_{ijkl}^2} = \frac{3! J^2}{N^3}$, which represent the disorder average.

Note. The correlation function results will be the ones after the disorder average was taken (Figure 2).

From the kinetic term, we can see that the energy dimensions are $[\chi_i] = 0$ and from the interaction one, $[J_{ijkl}] = 1$.

The free theory takes the form, of $J_{ijkl} = 0$ in 1:

$$\mathcal{L}_{free} = -\frac{1}{2} \sum_{i=1}^{N} \chi_i \partial_\tau \chi_i, \tag{2}$$

with a two-point function, shown in B

$$G_0 = \frac{1}{2}\, sgn(\tau) \tag{3}$$

(notice that there are some conventions of the 2-point function that use a minus sign in the definition).

For SYK, because of the disorder average, large N and the anti-commutation relation, the Feynman diagrams for the full (zero-temperature) 2-point function take a melonic form, as shown in Figure 1.

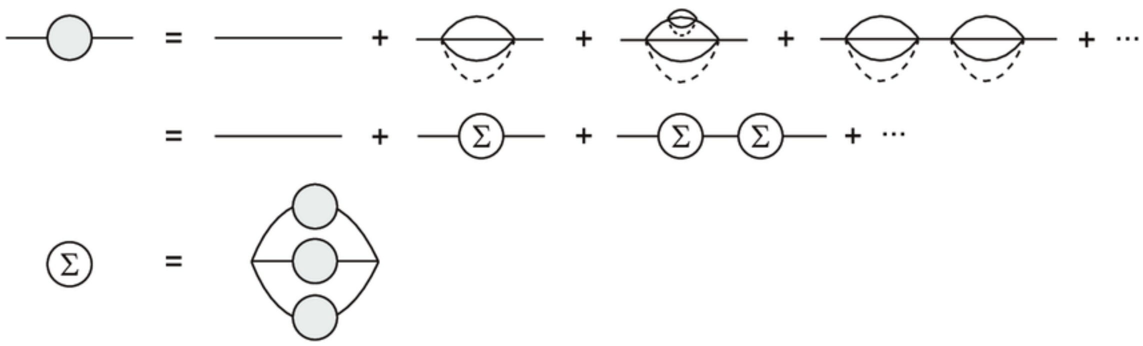


Figure 1: 2-point function as a sum of melon Feynman diagrams at large N (this sum contains only connected diagrams). The lines with a shaded circle correspond to a full dressed line and the dashed ones are to indicate the average disorder.

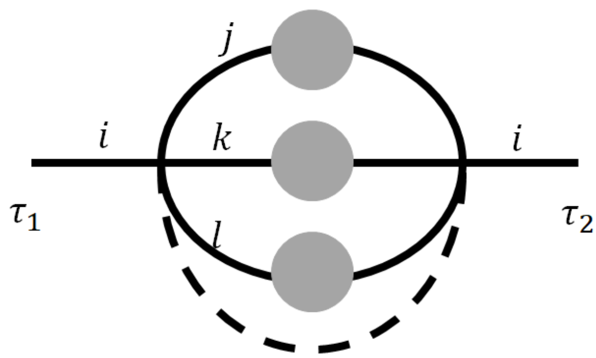


Figure 2: Form of a melonic SYK diagram after the disorder average was taken. The dashed line indicates the action of the average, as we can see from the matching of the indices.

It is easier to write such sum in Fourier space, therefore:

$$
\begin{aligned}
\tilde{G}(i\omega) &= \tilde{G}_0 + \tilde{G}_0\tilde{\Sigma}(i\omega)\tilde{G}_0 + \tilde{G}_0\tilde{\Sigma}(i\omega)\tilde{G}_0\tilde{\Sigma}(i\omega)\tilde{G}_0 + \dots \\
&= \tilde{G}_0 + \tilde{G}_0\tilde{\Sigma}(i\omega)\left(\tilde{G}_0 + \tilde{G}_0\tilde{\Sigma}(i\omega)\tilde{G}_0 + \tilde{G}_0\tilde{\Sigma}(i\omega)\tilde{G}_0\tilde{\Sigma}(i\omega)\tilde{G}_0 + \dots\right) \\
&= \tilde{G}_0 + \tilde{G}_0\tilde{\Sigma}(i\omega)\tilde{G}(i\omega) \\
\implies \tilde{G}(i\omega) &= \frac{1}{\tilde{G_0}^{-1} - \tilde{\Sigma}(i\omega)}
\end{aligned}
\tag{4}
$$

from appendix B, the dressed 2-point function is then,

$$
\tilde{G}(i\omega) = \frac{1}{-i\omega - \tilde{\Sigma}(i\omega)}. \tag{5}
$$

Known as Schwinger-Dyson equations. In coordinates space, we can write the corresponding integral form, as follows[2]:

$$
G(\tau) = G_0(\tau) + \int d\tau_1 \int d\tau_2 G_0(\tau - \tau_1)\Sigma(\tau_1 - \tau_2)G(\tau_2) \tag{6}
$$

and since we know $G_0(\tau) = \frac{1}{2}\ sgn(\tau) \implies \partial_\tau G(\tau) = \delta(\tau)$, therefore,

$$
\begin{cases}
\partial_\tau G(\tau) - \int d\tau' \Sigma(\tau - \tau')G(\tau') = \delta(\tau) \\
\Sigma(\tau) = J^2 G(\tau)^3
\end{cases}. \tag{7}
$$

The S-D equations can also be obtained by performing average disorder via the replica trick and evaluating the saddle point of the effective action. We start from the disordered average partition function:

$$
\begin{aligned}
\bar{Z} \equiv Z &= \int \mathcal{D}J_{ijkl} \int \mathcal{D}\chi_m e^{-\frac{J_{ijkl}^2 N^3}{2(3!J^2)}} e^{-\int d\tau \mathcal{L}(\tau)} \\
&= \int \mathcal{D}\chi_m \int \mathcal{D}J_{ijkl} e^{-\frac{J_{ijkl}^2 N^3}{2(3!J^2)} - \int d\tau \frac{1}{4!} J_{ijkl}\chi_i\chi_j\chi_k\chi_l - \frac{1}{2}\int d\tau \chi_i \partial_\tau \chi_i} \\
&= \int \mathcal{D}\chi_m \int \mathcal{D}J_{ijkl} e^{-\frac{J_{ijkl}^2 N^3}{2(3!J^2)} - \frac{1}{4!} J_{ijkl}\int d\tau \chi_i\chi_j\chi_k\chi_l + \frac{3!}{2(4!)}\frac{J^2}{N^3}\int d\tau_1 \int d\tau_2 \chi_i\chi_j\chi_k\chi_l(\tau_1)\chi_i\chi_j\chi_k\chi_l(\tau_2)} \\
&\qquad {}^{- \frac{3!}{2(4!)}\frac{J^2}{N^3}\int d\tau_1 \int d\tau_2 \chi_i\chi_j\chi_k\chi_l(\tau_1)\chi_i\chi_j\chi_k\chi_l(\tau_2) - \frac{1}{2}\int d\tau \chi_i \partial_\tau \chi_i} \\
&= \int \mathcal{D}\chi_m \int \mathcal{D}J_{ijkl} e^{-\left(\frac{J_{ijkl}}{J}\sqrt{\frac{N^3}{2(3!)}} + J\sqrt{\frac{1}{8N^3}}\int d\tau \chi_i\chi_j\chi_k\chi_l\right)^2 + \frac{J^2}{8N^3}\int d\tau_1 \int d\tau_2 \chi_i\chi_j\chi_k\chi_l(\tau_1)\chi_i\chi_j\chi_k\chi_l(\tau_2)} \\
&\qquad {}^{- \frac{1}{2}\int d\tau \chi_i \partial_\tau \chi_i} \\
&= \frac{J\sqrt{2(3!)\pi}}{\sqrt{N^3}} \int \mathcal{D}\chi_m e^{\frac{J^2}{8N^3}\int d\tau_1 \int d\tau_2 \chi_i\chi_j\chi_k\chi_l(\tau_1)\chi_i\chi_j\chi_k\chi_l(\tau_2) - \frac{1}{2}\int d\tau \chi_i \partial_\tau \chi_i}
\end{aligned}
\tag{8}
$$

Let, $G(\tau_1, \tau_2) = \frac{1}{N}\sum_i \chi_i(\tau_1)\chi_i(\tau_2)$ and $\Sigma(\tau_1, \tau_2) = 0$, therefore

[2] To show that $G(\tau_1, \tau_2) = G(\tau_1 - \tau_2)$, we use the definition and the fact that the energy of the vacuum is zero. So, $G_{ij}(\tau_1, \tau_2) = \langle \mathcal{T}\chi_i(\tau_1)\chi_j(\tau_2)\rangle = \left\langle \mathcal{T} e^{\tau_1 H}\chi_i(0)e^{-\tau_1 H}e^{\tau_2 H}\chi_j(0)e^{-\tau_2 H}\right\rangle = \left\langle \mathcal{T} e^{\tau_2 H}e^{(\tau_1-\tau_2)H}\chi_i(0)e^{-(\tau_1-\tau_2)H}\chi_j(0)e^{-\tau_2 H}\right\rangle = \left\langle \mathcal{T} e^{(\tau_1-\tau_2)H}\chi_i(0)e^{-(\tau_1-\tau_2)H}\chi_j(0)\right\rangle = \langle \mathcal{T}\chi_i(\tau_1 - \tau_2)\chi_j(0)\rangle = G(\tau_1 - \tau_2)$.

$$
\begin{aligned}
Z =& \frac{J\sqrt{2(3!)\pi}}{\sqrt{N^3}} \int \mathcal{D}\chi_m \int \mathcal{D}\Sigma\ \delta(N\Sigma) \int \mathcal{D}G\ \delta\left(G - \frac{1}{N}\chi_i\chi_i\right) \times \\
& \qquad \exp\left(\frac{J^2}{8N^3} \int d\tau_1 \int d\tau_2\ \chi_i\chi_j\chi_k\chi_l(\tau_1)\chi_i\chi_j\chi_k\chi_l(\tau_2) - \frac{1}{2}\int d\tau \chi_i \partial_\tau \chi_i\right) \\
=& \frac{J\sqrt{2(3!)\pi}}{\sqrt{N^3}} \int \mathcal{D}\chi_m \int \mathcal{D}\Sigma \int \mathcal{D}G \\
& \qquad \exp\left[-\frac{1}{2}\int d\tau_1 \int d\tau_2 N\Sigma(\tau_1,\tau_2)\left(G(\tau_1,\tau_2) - \frac{1}{N}\chi_i\chi_i\right)\right. \\
& \qquad \left. + \frac{NJ^2}{8N^4}\int d\tau_1 \int d\tau_2\ \chi_i\chi_j\chi_k\chi_l(\tau_1)\chi_i\chi_j\chi_k\chi_l(\tau_2) - \frac{1}{2}\int d\tau \chi_i \partial_\tau \chi_i\right] \\
=& \frac{J\sqrt{2(3!)\pi}}{\sqrt{N^3}} \int \mathcal{D}\chi_m \int \mathcal{D}\Sigma \int \mathcal{D}G \\
& \qquad \exp\left[-\frac{1}{2}\int d\tau_1 \int d\tau_2 \left(N\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \Sigma(\tau_1,\tau_2)\chi_i(\tau_1)\chi_i(\tau_2)\right.\right. \\
& \qquad \left.\left. - \frac{NJ^2}{4}G(\tau_1,\tau_2)^4\right) - \frac{1}{2}\int d\tau \chi_i \partial_\tau \chi_i\right] \\
=& \frac{J\sqrt{2(3!)\pi}}{\sqrt{N^3}} \int \mathcal{D}\Sigma \int \mathcal{D}G\ \det\left(\partial_\tau - \Sigma\right)^{N/2} \times \\
& \qquad \exp\left[-\frac{N}{2}\int d\tau_1 \int d\tau_2 \left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{4}G(\tau_1,\tau_2)^4\right)\right] \\
=& J\sqrt{\frac{12\pi}{N^3}} \int \mathcal{D}\Sigma \int \mathcal{D}G\ e^{-N\frac{I_{eff}}{N}},
\end{aligned} \tag{9}
$$

where

$$
I_{SYK} \equiv \frac{I_{eff}}{N} = -\frac{1}{2}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] + \frac{1}{2}\int d\tau_1 \int d\tau_2 \left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{4}G(\tau_1,\tau_2)^4\right), \tag{10}
$$

here N plays the role of $\hbar^{-1}$, the saddle of the action at leading order when $\hbar \to 0$ gives us the Euler-Lagrange equations. Hence, if we look for the saddle point of this effective action at leading order in $\frac{1}{N}$, for $N \gg 1$ (semi-classical limit), we have:

$$
\begin{aligned}
& \left.\frac{\delta I_{eff}}{\delta \Sigma}\right|_{\Sigma=\tilde{\Sigma}} = 0 \\
& \implies \tilde{G(\tau)} = \frac{1}{\partial_\tau - \tilde{\Sigma}(\tau)} \\
& \left.\frac{\delta I_{eff}}{\delta G}\right|_{G=\tilde{G}} = 0 \\
& \implies \tilde{\Sigma}(\tau) = J^2 \tilde{G}(\tau)^3,
\end{aligned} \tag{11}
$$

which are the SD equations.

Note. *Where does the replica trick*[3] *appear in this calculation?*

The physical thermodynamical quantity here is $F = -\frac{1}{\beta}\log Z$*, so performing the disorder average to* F*:* $\overline{F} = -\frac{1}{\beta}\overline{\log Z}$*. However, by the replica trick, we can argue (in SYK) that* $\overline{F} = -\frac{1}{\beta}\log \overline{Z}$*.*

The difference between $\overline{\log Z}$ *and* $\log \overline{Z}$ *is of order* N^{2-q}*, for* SYK_q *(namely, for* $q = 4$*, the replica off-diagonal terms are subleading in comparison with the diagonal ones. So, we can use* $\overline{\log Z}$ *and* $\log \overline{Z}$ *interchangeably).*

Strictly speaking, the derivation follows from (taken from Kitaev and Su, The Soft Mode in the SYK model):

$$\overline{Z^n} = \int \mathcal{D}\Sigma\mathcal{D}G\ e^{-I^{(n)}[\Sigma,G]} \approx \exp\left[-\min_{\Sigma}\ \max_{G}\ I^{(n)}[\Sigma, G]\right],$$

where

$$I^{(n)}[\Sigma, G] = -\frac{N}{2}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] + \frac{N}{2}\sum_{\alpha,\beta=1}^{n}\int d\tau d\tau' \left(\Sigma_{\alpha\beta}(\tau,\tau')G_{\alpha\beta}(\tau,\tau') - \frac{J^2}{4}G_{\alpha\beta}(\tau,\tau')^4\right)$$

with α *and* β *being the replica indices. In the large* N *limit, the outer integrals* $\mathcal{D}\Sigma\mathcal{D}G$ *can be performed by finding the saddle point: Maximum at* G *and minimum at* Σ *(classical solutions).*

The most natural solution for the minimum over Σ *is diagonal over replicas*

$$\Sigma_{\alpha\beta}(\tau,\tau') = \Sigma(\tau,\tau')\delta_{\alpha\beta}.$$

With such ansatz, taking the $n \to 0$ *limit, we obtain* $\overline{\log Z} \approx \log \overline{Z} \approx -I_{eff}[\tilde{\Sigma}, \tilde{G}]$*, as argued before.*

There is a subtlety in the previous action, which is that we need to regularize the determinant term to eliminate the UV divergence, viz

$$\det\left(\partial_\tau - \Sigma\right) \to \frac{\det\left(\partial_\tau - \Sigma\right)}{\det\left(\partial_\tau\right)}.$$

The self-consistency of SD equations (7) (or similar ones) can be written for any model with all-to-all interaction. BUT, their solution may not be physical if some ordering occurs, such as spin glasses (see appendix C). For the $q = 4$ SYK model, the transition to a glassy phase is expected at extremely low temperature: $T_{glass} \sim Je^{-\sqrt{N}}$ [Quantum Fluctuations, Sachdev]. Therefore, we assume that $T \gg T_{glass}$, in this regime the mean field solution is accurate if $T \gg \frac{J}{N}$ (lower temperatures imply that quantum fluctuations must be taken into account).

Besides, in order to have a holographic model, it is important that there is not a spin glass phase. Why? Because a maximal Lyapunov exponent could most likely occur in the quantum

[3]The replica trick states that in quenched disordered systems and spin glasses,

$$\log Z = \lim_{n\to 0}\frac{Z^n - 1}{n},$$

that is, the average over n copies of the system. The subtlety in this method is that n is assumed to be an integer, but to recover $\log Z$, n has to approach zero continuously (this is not solved yet).

regime (at low temperatures). Hence, it is important that the system does not freeze into a spin glass as T is lowered[4].

Let's turn our attention in the IR limit of the SD equations, i.e. at low energies (or strong coupling, $J|\tau| \gg 1$). Here we drop the $\partial_\tau G(\tau)$ term and we know a solution of SD (see appendix D):

$$G_{IR}(\tau) = \left(\frac{1}{4\pi J^2}\right)^{1/4} \frac{1}{\sqrt{|\tau|}}\, sgn(\tau) \tag{12}$$

The new set of SD equations have an additional reparametrization symmetry in time. If $\tau \to f(\tau)$, then $G(\tau_1, \tau_2) \to |f'(\tau_1)f'(\tau_2)|^{1/q} G(f(\tau_1), f(\tau_2))$:

$$\begin{aligned} J^2 \int d(f(\tau'))\ \Sigma(f(\tau), f(\tau'))G(f(\tau'), f(0)) &= J^2 \int d\tau' |f'(\tau')|\ G^3(f(\tau), f(\tau'))G(f(\tau'), f(0)) \\ &= J^2 \int d\tau' |f'(\tau')|\ \frac{G^3(\tau, \tau')}{|f'(\tau')f'(\tau)|^{3/4}} \frac{G(\tau', 0)}{|f'(\tau')f'(0)|^{1/4}} \\ &= J^2 \int d\tau'\ \frac{G^3(\tau, \tau')G(\tau', 0)}{|f'(\tau)|^{3/4}|f'(0)|^{1/4}} \\ &= -\frac{\delta(\tau, 0)}{|f'(0)|} = -\delta(f(\tau), f(0)), \end{aligned} \tag{13}$$

where in the last line, we used the SD equation in the IR limit. Thus, we can see that, at low energies, the theory is invariant under the reparametrization group. Furthermore, if we map the time line into a thermal circle (compactify time) and choosing $f(\tau) = \tan \frac{\pi\tau}{\beta}$, then

$$\begin{aligned} G^\beta_{IR} &= \left(\frac{1}{4\pi J^2}\right)^{1/4} \left(\frac{\pi}{\beta}\right)^{2/4} \frac{sgn(\tau)}{\sqrt{\left|\tan\left(\frac{\pi\tau}{\beta}\right)\right|}} \left|\sec^2\left(\frac{\pi\tau}{\beta}\right)\right|^{1/4} \\ \implies G^\beta_{IR} &= \frac{\pi^{1/4}}{\sqrt{2}\sqrt{\beta J}} \frac{sgn(\tau)}{\sqrt{\left|\sin\left(\frac{\pi\tau}{\beta}\right)\right|}}. \end{aligned} \tag{14}$$

This is the non-zero temperature 2-point function (in the IR).

Let us comment a little more about the IR emergent reparametrization symmetry: first, in one-dimension it is equal to the conformal invariance, and second, in this limit we have a space of solutions, namely, moving among those solutions has no energy cost (the action does not change if we perform such transformation). Therefore, we will find divergences in the correlation functions (as in the 2-point function at $\tau = 0$ or $\tau = \beta$). Besides, choosing a particular solution from the saddle point equations implies a spontaneously breaking of the symmetry.

[4] In this sense, SYK is simpler than SY, for SYK requires only one scaling limit $N \gg 1$, whereas SY requires two $N \gg 1$ and $M \gg 1$, where

$$H_{SY} = \frac{1}{\sqrt{M}} \sum_{j,k=1}^{N} J_{jk} \vec{S}_j \cdot \vec{S}_k, \qquad P(J_{jk}) \sim \exp\left(-\frac{J_{jk}^2}{2J^2}\right)$$

and the spins are in some representation of $SU(M)$. So, depending on such representation, the ground state may or may not have a spin glass phase.

How do we deal with this? Well, we have to recall that this is a limiting behavior, the SYK model does not have reparametrization symmetry (broken by the UV term) and, consequently, those divergences do not exist. Furthermore, this symmetry is broken explicitly (not spontaneously, that only works in the range of very low energies).

However, we would like to stay as close as possible to the IR. So, let's see what is the cost of the reparametrization transformation close to the IR point. To do this, let $\sigma(\tau_1, \tau_2) = \delta(\tau_1, \tau_2)\partial_{\tau_1}$, so that the effective action (10) writes:

$$\frac{I_{eff}}{N} = -\frac{1}{2}\log\left[\det\left(\sigma - \Sigma\right)\right] + \frac{1}{2}\int d\tau_1 \int d\tau_2 \left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{4}G(\tau_1,\tau_2)^4\right),$$

and perform the change of variables $\Sigma \to \Sigma + \sigma$, then

$$\begin{aligned}\frac{I_{eff}}{N} = -\frac{1}{2}\log\left[\det\left(-\Sigma\right)\right] + \frac{1}{2}\int d\tau_1 \int d\tau_2 \left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{4}G(\tau_1,\tau_2)^4\right) + \\ + \frac{1}{2}\int d\tau_1 \int d\tau_2 \; \sigma(\tau_1,\tau_2)G(\tau_1,\tau_2),\end{aligned}$$

where the last term is not invariant under reparametrization of time (can be thought of as a perturbation of the IR action). If we are close to the IR, then our goal is to expand the last term perturbatively to get the first non-zero correction to the CFT previously found.

Start with the low-energies solution (12):

$$\begin{aligned}G(\tau_1,\tau_2) &= b\, \frac{sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \\ &\to b\; sgn(\tau_{12})\frac{|f'(\tau_1)f'(\tau_2)|^{\Delta}}{J^{2\Delta}\,|f(\tau_1) - f(\tau_2)|^{2\Delta}},\end{aligned}$$

where $\tau_{12} = \tau_1 - \tau_2$ and $\Delta = 1/q$. Now let $\tau_+ = \frac{\tau_1+\tau_2}{2}$ and expand f around τ_+:

$$f(\tau_1) = f\left(\tau_+ + \frac{\tau_{12}}{2}\right) = f(\tau_+) + \frac{\tau_{12}}{2}f'(\tau_+) + \frac{\tau_{12}^2}{4}\frac{f''(\tau_+)}{2} + \frac{\tau_{12}^3}{8}\frac{f'''(\tau_+)}{3!} + \cdots$$

$$f(\tau_2) = f\left(\tau_+ - \frac{\tau_{12}}{2}\right) = f(\tau_+) - \frac{\tau_{12}}{2}f'(\tau_+) + \frac{\tau_{12}^2}{4}\frac{f''(\tau_+)}{2} - \frac{\tau_{12}^3}{8}\frac{f'''(\tau_+)}{3!} + \cdots$$

$$f'(\tau_1) = f'\left(\tau_+ + \frac{\tau_{12}}{2}\right) = f'(\tau_+) + \frac{\tau_{12}}{2}f''(\tau_+) + \frac{\tau_{12}^2}{4}\frac{f'''(\tau_+)}{2} + \cdots$$

$$f'(\tau_2) = f'\left(\tau_+ - \frac{\tau_{12}}{2}\right) = f'(\tau_+) - \frac{\tau_{12}}{2}f''(\tau_+) + \frac{\tau_{12}^2}{4}\frac{f'''(\tau_+)}{2} + \cdots$$

Therefore,

$$
\begin{aligned}
G(\tau_1,\tau_2) &\approx b\ sgn(\tau_{12}) \frac{|f'(\tau_1)f'(\tau_2)|^{\Delta}}{J^{2\Delta}\left|\tau_{12}f'(\tau_+)+\frac{\tau_{12}^3}{4}\frac{f'''(\tau_+)}{3!}\right|^{2\Delta}} \\
&\approx b\ \frac{sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \frac{\left|f'(\tau_+)^2+\frac{\tau_{12}^2}{4}f'(\tau_+)f'''(\tau_+)-\frac{\tau_{12}^2}{4}f''(\tau_+)^2\right|^{\Delta}}{\left|f'(\tau_+)+\frac{\tau_{12}^2}{4}\frac{f'''(\tau_+)}{3!}\right|^{2\Delta}} \\
&\approx b\ \frac{sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \frac{\left|1+\frac{\tau_{12}^2}{4}\frac{f'''(\tau_+)}{f'(\tau_+)}-\frac{\tau_{12}^2}{4}\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2\right|^{\Delta}}{\left|1+\frac{\tau_{12}^2}{4}\frac{f'''(\tau_+)}{3!f'(\tau_+)}\right|^{2\Delta}} \\
&\approx b\ \frac{sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \left(1+\Delta\frac{\tau_{12}^2}{4}\left[\frac{f'''(\tau_+)}{f'(\tau_+)}-\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2\right]\right)\left(1-2\Delta\frac{\tau_{12}^2}{4}\frac{f'''(\tau_+)}{3!f'(\tau_+)}\right) \\
&\approx b\ \frac{sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \left(1+\Delta\frac{\tau_{12}^2}{4}\left[\frac{f'''(\tau_+)}{f'(\tau_+)}-\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2\right]-\Delta\frac{\tau_{12}^2}{2}\frac{f'''(\tau_+)}{3!f'(\tau_+)}\right) \\
&\approx b\ \frac{sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \left(1+\Delta\frac{\tau_{12}^2}{4}\left[\frac{f'''(\tau_+)}{f'(\tau_+)}-\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2-\frac{f'''(\tau_+)}{3f'(\tau_+)}\right]\right) \\
&\approx b\ \frac{sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \left(1+\Delta\frac{\tau_{12}^2}{4}\left[\frac{2}{3}\frac{f'''(\tau_+)}{f'(\tau_+)}-\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2\right]\right) \\
&\approx b\ \frac{sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \left(1+\Delta\frac{\tau_{12}^2}{6}\left[\frac{f'''(\tau_+)}{f'(\tau_+)}-\frac{3}{2}\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2\right]\right) \\
&\approx b\ \frac{sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \left(1+\frac{\Delta}{6}\ \tau_{12}^2\ Sch\left(f(\tau_+),\tau_+\right)\right),
\end{aligned}
$$

where $Sch\left(f(\tau),\tau\right)=\frac{f'''(\tau)}{f'(\tau)}-\frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2$. Hence, close to the IR and under reparametrization of time, we have an additional contribution to the action that reads

$$
I_{Sch}\approx b\ \frac{\Delta}{12J^{2\Delta}}\int d\tau_1\int d\tau_2\ \sigma(\tau_1,\tau_2)\frac{sgn(\tau_{12})}{|\tau_{12}|^{2\Delta-2}}\ Sch\left(f(\tau_+),\tau_+\right),
$$

this is called the Soft mode contribution to the action described by the Schwarzian. Why soft? The reason is that, at the IR, the reparametrization of time IS a symmetry and we have a family of solutions that describe the system. But, close to the low energy limit, the system has only one minima (the reparametrization symmetry is broken) and thus I_{Sch} describes how the system is slowly oscillating around it.

3 Four-point function

The 4-point function is given by

$$
\Gamma(\tau_1,\tau_2,\tau_3,\tau_4)=\frac{1}{N^2}\sum_{i,j=1}^{N}\langle\chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\rangle=G(\tau_1,\tau_2)G(\tau_3,\tau_4)+\frac{1}{N}\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4)+\cdots.
$$

Where the connected diagrams $\mathcal{F}$, at large N, are ladder diagrams[5] and scale as $\frac{1}{N}$:

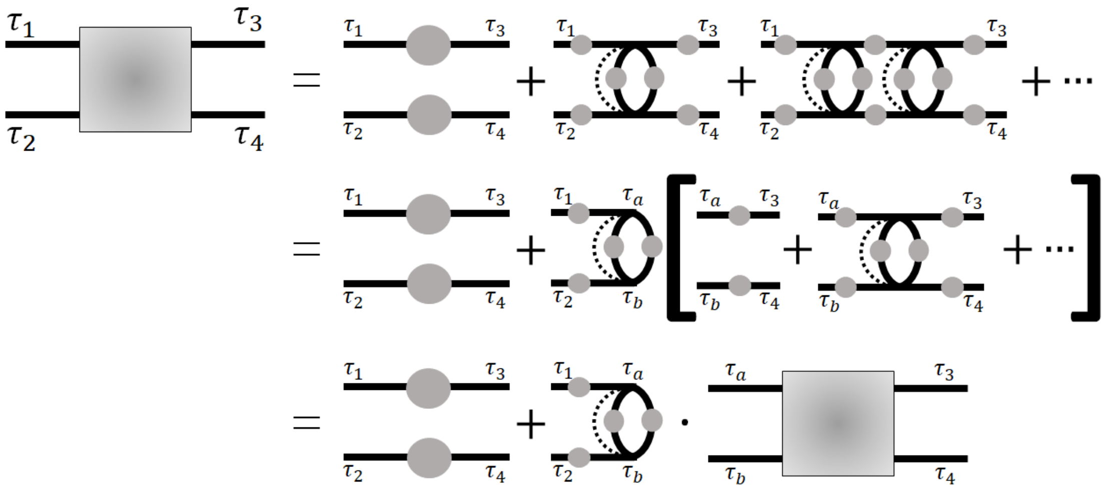


Figure 3: 4-point function as a sum of ladder diagrams at large N.

Therefore, the $\frac{1}{N}$ contribution to the four-point function can be read off from the previous diagrams as:

$$\mathcal{F}(\tau_1, \tau_2, \tau_3, \tau_4) = \mathcal{F}_0(\tau_1, \tau_2, \tau_3, \tau_4) + \int d\tau_a \int d\tau_b \; K(\tau_1, \tau_2, \tau_a, \tau_b)\mathcal{F}(\tau_a, \tau_b, \tau_3, \tau_4), \tag{15}$$

where[6]

$$\begin{aligned} \mathcal{F}_0(\tau_1, \tau_2, \tau_3, \tau_4) &= G(\tau_1 - \tau_4)G(\tau_2 - \tau_3) - G(\tau_1 - \tau_3)G(\tau_2 - \tau_4) \\ K(\tau_1, \tau_2, \tau_a, \tau_b) &= -3J^2 G(\tau_1 - \tau_a)G(\tau_2 - \tau_b)G^2(\tau_a - \tau_b) \end{aligned} \tag{16}$$

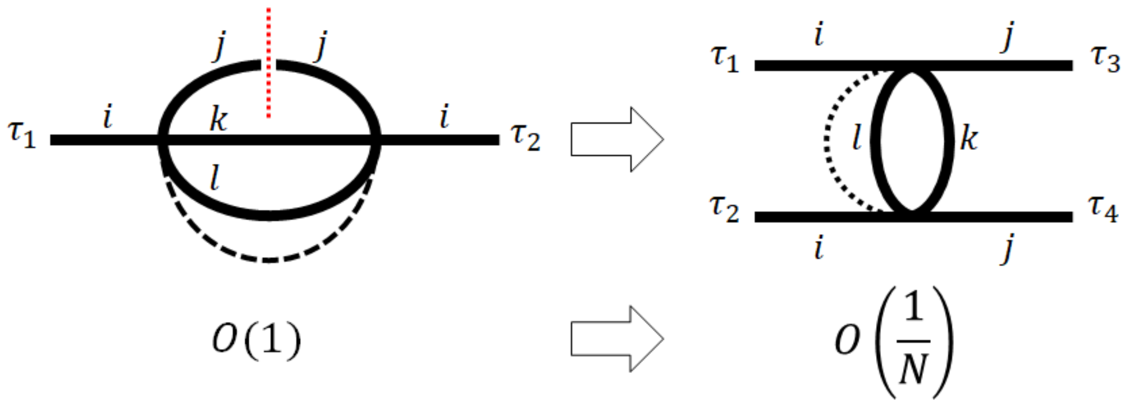


Figure 4: Cut of a melon diagram is equivalent to a ladder diagram that contributes to the four-point function. The cut of a line correspond to a division by N.

[5]Naively, as one can see this in Figure 4, a cut of one of the lines of the leading melon contributions is topologically equivalent to a ladder diagram that scales as $\frac{1}{N}$ since there is no sum over such line anymore (division by N).

[6]The minus signs come from the definition of Γ by moving one fermion and applying the anti-commutation relation. The factor of three comes from the interaction $\propto 3!$, but by symmetry we can exchange the dummy times τ_a and τ_b, which implies a division by 2 for not overcounting (from the Feynman diagrams point of view, this can be argued as the symmetric factor obtained by exchanging the two loop lines).

One way of solving $\mathcal{F}$ through (15) is to start with a set of eigenvectors:

$$v_\alpha(\tau_a, \tau_b) = \frac{1}{|\tau_a - \tau_b|^{2\alpha}} \; sgn(\tau_a - \tau_b), \tag{17}$$

with eigenvalues[7]

$$g(\alpha) = -\frac{3}{2}\frac{1}{(1-2\alpha)\tan(\pi\alpha)}. \tag{18}$$

With this set of eigenvectors, one can use the $SL(2,\mathbb{R})$ algebra to generate all of the eigenvectors. The key property is that conformal invariance allows to diagonalize K.

Another way of finding $\mathcal{F}$ is: the sum in Figure 3 can also be seen as a geometric series, where the kernel K is the ratio of such series[8], namely,

$$\mathcal{F} = \sum_n K^n \mathcal{F}_0 = \frac{1}{1-K}\mathcal{F}_0, \tag{19}$$

where in the last equality it is intended as matrix inversion.

A way of solving this equation is summarized in the following steps:

1) Understand properties of $\mathcal{F}$ as a function of the cross ratio.

2) Find the eigenfunctions of the Casimir $\mathcal{C}$ of the group $SL(2,\mathbb{R})$ with the previous properties.

3) Determine the set of eigenvalues h of $\mathcal{C}$ to have a complete basis of functions.

4) Compute the eigenvalues $k(h)$ of K as a function of h.

5) Calculate the inner products $\langle \Psi_h, \mathcal{F}_0 \rangle$ and $\langle \Psi_h, \Psi_h \rangle$

6) Obtain $\mathcal{F}(z) = \frac{1}{1-K}\mathcal{F}_0 = \sum_h \Psi_h(z) \frac{1}{1-k(h)} \frac{\langle \Psi_h, \mathcal{F}_0 \rangle}{\langle \Psi_h, \Psi_h \rangle}$.

In the conformal limit, $\mathcal{F}$ will transform under $SL(2,\mathbb{R})$ like a four-point function of fields of dimension Δ:

$$\mathcal{F}(\tau_1, \tau_2, \tau_3, \tau_4) = G(\tau_{12})G(\tau_{34})\mathcal{F}(z), \qquad z = \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}, \tag{20}$$

i.e.

$$\langle \chi(\tau_1)\chi(\tau_2)\chi(\tau_3)\chi(\tau_4) \rangle = \langle \chi(\tau_1)\chi(\tau_2) \rangle \langle \chi(\tau_3)\chi(\tau_4) \rangle \sum_h (C^h_{\chi\chi})^2 z^h \; {}_2F_1(h, h, 2h, z),$$

where

- h runs over the set of conformal primaries (scaling dimensions of primaries).

- $C^h_{\chi\chi}$ set of 3-point function coefficients (or Operator Product Expansion, OPE, coefficients).

[7]The proof that this set of functions are eigenfunctions of K is shown in Appendix E.

[8]Here, we treat $\mathcal{F}$, $\mathcal{F}_0$ and K as matrices whose one index is formed by the first two arguments and the other index by the last two.

- $z^h\ {}_2F_1(h,h,2h,z)$ is the conformal block, summing the $SL(2,\mathbb{R})$ contribution of the descendants of the primary h.

The above expression is called the *decomposition into conformal blocks*: Naively, it is the same as inserting a complete set of states between the two pairs of operators. Succeeding in writing $\mathcal{F}$ in such a form, the scaling dimensions h and OPE coefficients can be read off explicitly.

Therefore, let us use the following generators of the $SL(2,\mathbb{R})$ algebra:

$$\begin{aligned}
\hat{D} &= -\tau\partial_\tau - \Delta, & \hat{P} &= \partial_\tau, & \hat{K} &= \tau^2\partial_\tau + 2\tau\Delta \\
[\hat{D},\hat{P}] &= \hat{P}, & [\hat{D},\hat{K}] &= -\hat{K}, & [\hat{P},\hat{K}] &= -2\hat{D}
\end{aligned}$$

One can check that all this generators commute with the kernel, in the following sense[9]:

$$(\hat{P}_1+\hat{P}_2)K(\tau_1,\tau_2,\tau_3,\tau_4) = K(\tau_1,\tau_2,\tau_3,\tau_4)(\hat{P}_3+\hat{P}_4)$$

Therefore, this symmetry implies that the 4-point functions are simple powers times a function of the $SL(2,\mathbb{R})$ invariant cross-ratio: z. So, we can represent the kernel in the space of functions of a single cross-ratio instead of the space of functions of 2 times, viz, $K(z,\tilde{z})$ instead of $K(\tau_1,\tau_2,\tau_3,\tau_4)$.

Now, since K commutes with the generators of the $SL(2,\mathbb{R})$ algebra, it commutes with the Casimir operator $\mathcal{C}$ as well, where $\mathcal{C}$ is built from the sum of the generators acting on 2 times:

$$\begin{aligned}
\mathcal{C} &= (\hat{D}_1+\hat{D}_2)^2 - \frac{1}{2}(\hat{K}_1+\hat{K}_2)(\hat{P}_1+\hat{P}_2) - \frac{1}{2}(\hat{P}_1+\hat{P}_2)(\hat{K}_1+\hat{K}_2) \\
&= 2(\Delta^2-\Delta) - \hat{K}_1\hat{P}_2 - \hat{P}_1\hat{K}_2 + 2\hat{D}_1\hat{D}_2
\end{aligned}$$

for the second line, we used:

$$\begin{aligned}
\hat{D}^2 &= (-\tau\partial_\tau-\Delta)(-\tau\partial_\tau-\Delta) = \tau\partial_\tau + \tau^2\partial_\tau^2 + 2\Delta\tau\partial_\tau + \Delta^2 \\
\frac{1}{2}\hat{P}\hat{K} &= \frac{1}{2}\partial_\tau(\tau^2\partial_\tau + 2\tau\Delta) = \tau\partial_\tau + \frac{\tau^2}{2}\partial_\tau^2 + \Delta + \tau\Delta\partial_\tau \\
\frac{1}{2}\hat{K}\hat{P} &= \frac{1}{2}(\tau^2\partial_\tau + 2\tau\Delta)\partial_\tau = \frac{\tau^2}{2}\partial_\tau^2 + \tau\Delta\partial_\tau.
\end{aligned}$$

Why do we care about the Casimir? Simply because it is easier to diagonalize. Since it is a differential operator with a family of eigenfunctions given by simple powers times functions $\Psi_h(z)$ and because of the non-degeneracy of the spectrum, $\Psi_h(z)$ must be the eigenfunctions of $K(z,\tilde{z})$.

$\mathcal{C}$ is an order 2 operator, so if $\mathcal{C}\Psi_h = h(h-1)\Psi_h$, then $K\Psi_h = k(h)\Psi_h$. This equation has multiple solutions for any $h\in\mathbb{C}$. Thus, the general solution to $\mathcal{C}\Psi_h = h(h-1)\Psi_h$ is a linear combination of conformally invariant three-point functions:

$$\Psi_h(\tau_1,\tau_2) = \int d\tau_0\ g_h(\tau_0) f_h^{\tau_0}(\tau_1,\tau_2), \tag{21}$$

where $f_h^{\tau_0}(\tau_1,\tau_2) = \frac{sgn(\tau_{12})}{|\tau_{01}|^h|\tau_{02}|^h|\tau_{12}|^{1-h}}$. By replacing this in $K\Psi_h = k(h)\Psi_h$, we can obtain[10] $k(h)$:

[9] Here the notation $\hat{P}_i = \partial_{\tau_i}$

[10] In general, the eigenfunctions of the exact (thermal) kernel depend on two numbers:

$$W_{h,m}(\tau_1,\tau_2) = \frac{1}{\beta}\int_0^\beta d\tau_0\ W_h^{\tau_0}(\tau_1,\tau_2)e^{im\frac{2\pi\tau_0}{\beta}},$$

$$k(h) = -\frac{3}{2}\frac{\tan\left[\frac{\pi}{2}\left(h-\frac{1}{2}\right)\right]}{h-\frac{1}{2}} \tag{22}$$

If we replace $h = 2\alpha - \frac{1}{2}$, then we obtain $k\left(2\alpha - \frac{1}{2}\right) = g(\alpha)$.

However, from the solutions $\Psi_h(\tau_1, \tau_2)$, we need to find the subset that form a complete basis of anti-symmetric eigenfunctions for a suitable choice of inner product (this is a non-trivial calculation, we can refer to Maldacena and Stanford paper). In other words, this is to say that we restrict the eigenvalue of the Casimir $h(h-1)$ has to be real. So, the allowed values of h are:

$$h_s = \frac{1}{2} + is, \;\; s \in \mathbb{R} \qquad \& \qquad h_n = 2n \;\; n \in \mathbb{Z}^+.$$

This allow us to write

$$\mathcal{F}(z) = \int_{-\infty}^{\infty} ds \, \frac{\alpha_{h_s}(\tau_1, \ldots, \tau_4)}{1 - k(h_s)} \Psi_h(z) + \sum_{n=1}^{\infty} \frac{\beta_{h_n}(\tau_1, \ldots, \tau_4)}{1 - k(h_n)} \Psi_h(z), \tag{23}$$

where α and β are functions that depend on the eigenfunctions and measure factors coming from the inner product.

An important fact here is the $n = 1$ case, i.e. at $h_1 = 2$, we obtain

$$k(h_1) = k(2) = -\frac{3}{2}\frac{\tan\left(\frac{3\pi}{4}\right)}{\frac{3}{2}} = 1.$$

Clearly, this represents a divergence in the four-point function[11]. Although, neglecting such pole, from (23) we need to extract the scaling dimensions and the OPE coefficients. To do this, the claim is that β_h is such that

$$\mathcal{F}(z) = \int_{-\infty}^{\infty} ds \, \frac{\alpha_{h_s}(\tau_1, \ldots, \tau_4)}{1 - k(h_s)} \Psi_h(z) + \sum_{n=2}^{\infty} \text{Res}\left[\frac{\alpha_{h_n}(\tau_1, \ldots, \tau_4)}{1 - k(h_n)} \Psi_h(z)\right]\Bigg|_{h=2n}, \tag{24}$$

and this can be interpreted as a single contour integral whose contour is a line and small circles around the poles of α_{h_n} (at $h = 2n$), as shown in Figure 5. However, the poles at even integer values are cancelled by deforming the contour by pushing the continuous line to the right, at the cost of picking up the poles of $\frac{1}{1-k(h)}$, namely, the values h_m for which $k(h_m) = 1$.

where $W_h^{\tau_0}(\tau_1, \tau_2) = \frac{sgn(\tau_{12})}{|\sin\frac{2\pi\tau_{01}}{\beta}|^h |\sin\frac{2\pi\tau_{02}}{\beta}|^h |\sin\frac{2\pi\tau_{12}}{\beta}|^{1-h}}$. But, in the conformal limit these functions are all degenerate in m. In the zero-T case described above, the dependence on m is included in the function $g_h(\tau_0)$.

[11]It is important to notice here that this pole arises from the fact that we are using the conformal kernel (IR limit), if we use the exact kernel (full spectrum of energies) we will find a correction so that $k(h_1) = 1 + \#$, which is a big number, but not a pole.

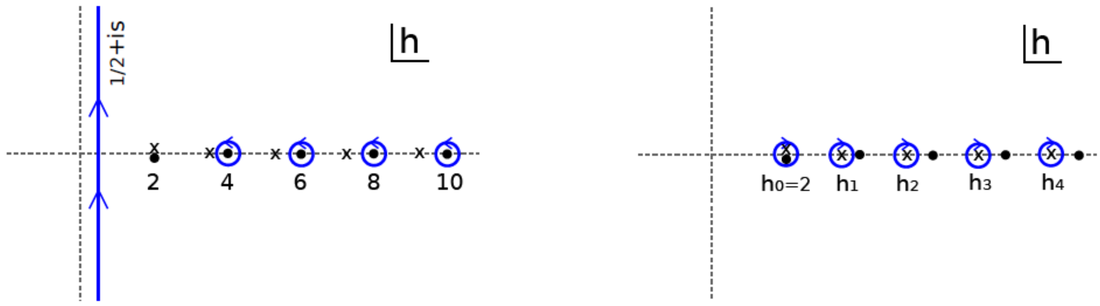


Figure 5: Contour of integration of $\mathcal{F}$. On the left it is a line and small circles enclosing the poles of α_h. On the right, the line integration contour is moved in such a way that it now cancels the previous poles of α_h (different orientations) and encloses the values of h such that $k(h) = 1$. Image taken from Maldacena and Stanford paper.

Therefore, the new sum of residues turns out to be (without the $h_0 = 2$ pole):

$$\mathcal{F}(z) = \sum_{m=1}^{\infty} c_{h_m}^2 z^{h_m} \ {}_2F_1(h_m, h_m, 2h_m, z), \tag{25}$$

where we have written the 4-point function in a conformal block way, with scaling dimensions $h_m = 2\Delta + 1 + 2m + \epsilon_m$ (ϵ_m are the anomalous dimensions) and the OPE coefficients are proportional to c_{h_m}.

We can write the generalization of the eigenvalues of the kernel, since the previous formula is valid for $q = 4$. For any q, the corresponding expression is

$$k(h, q) = -(q-1)\frac{\Gamma\left(\frac{3}{2} - \frac{1}{q}\right)\Gamma\left(1 - \frac{1}{q}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{q}\right)\Gamma\left(\frac{1}{q}\right)}\frac{\Gamma\left(\frac{1}{q} + \frac{h}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{1}{q} - \frac{h}{2}\right)}\frac{\Gamma\left(\frac{1}{2} + \frac{1}{q} - \frac{h}{2}\right)}{\Gamma\left(1 - \frac{1}{q} + \frac{h}{2}\right)}. \tag{26}$$

From this we can see that $k(h, 2) = -1$ (independent of h), so SYK_2 does not have any poles in the kernel.

As discussed before, the solutions to $k(h, q) = 1$ correspond to divergences of the (conformal) kernel. Let us focus on the $h_1 = 3 + \frac{2}{q} + \epsilon_1$ mode. Considering $q \gg 1$, and expanding (26), we have[12] up to $O(1/q^4)$:

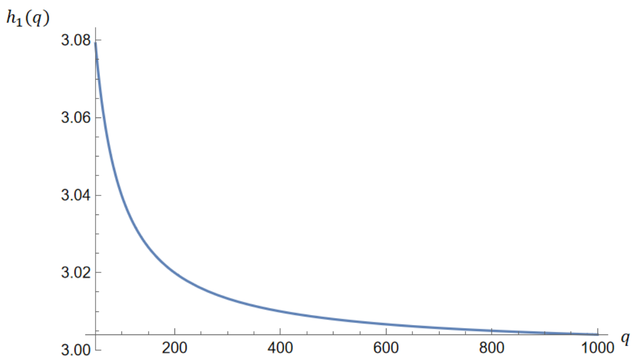
$$h_1 = 3 + \frac{4}{q} - \frac{2}{q^2} + \frac{4\pi^2 - 47}{q^3} + \frac{\left(\frac{797}{2} - \frac{52\pi^2}{3} - 192\zeta(3)\right)}{q^4} + \cdots \tag{27}$$

Similarly, solving numerically for small q, we can see the results in Figure 6. From which we can extract the first 3 non-integer values of h_m for $q = 4$, namely

$$h_1(4) \approx 3.773535618637616, \qquad h_2(4) \approx 5.679458989211242, \qquad h_3(4) \approx 7.631970759040549.$$

that is, the canonical dimension of the primary operator $\mathcal{O}_{h_1}$ in Majorana SYK is 3.5 and its corresponding anomalous dimension is $\epsilon_1 \approx 0.2735356186376161\ldots$

[12] In order to fix the coefficients of the expansion, we start with $h_1 = 3 + \frac{\alpha}{q}$, solve $k(h_1, q) = 1$ and find the value of x by requiring the leading order coefficient to be 1 ($\alpha = 4$). Then, we write $h_1 = 3 + \frac{4}{q} + \frac{\beta}{q^2}$ and expand (26) up to $1/q$ order, fix the coefficient to be zero and find $\beta = -2$. The following coefficients are fixed in a similar way, i.e. by asking the coefficients of higher orders in the $1/q$ expansion, of the equation $k(h_1, q) = 1$, to be zero.



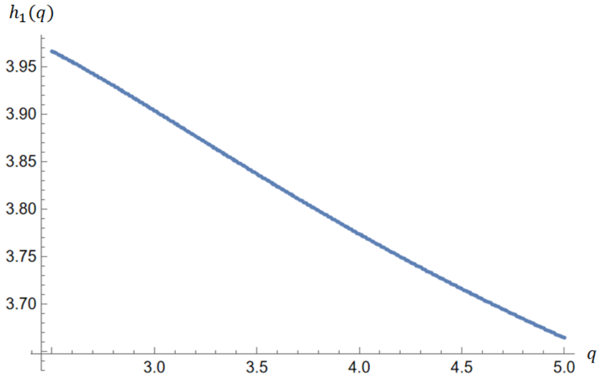


Figure 6: Kernel eigenvalues for large q (left) and small q (right), obtained by solving equation (26) for the h_1 mode.

Note. *How can one see that the 4-point function is not divergent in the $h_0 = 2$ mode?*

To answer this question, let us recall what limiting values the (zero-T) 2-point function has:

$$G(\tau) = \begin{cases} \left(\frac{1}{4\pi}\right)^{1/4} \frac{sgn(\tau)}{\sqrt{|J\tau|}} & , \quad J|\tau| \gg 1 \quad (IR) \\ \frac{1}{2}\, sgn(\tau) & , \quad J|\tau| \ll 1 \quad (UV) \end{cases} . \tag{28}$$

The four-point function previously obtained was done considering the conformal (IR) solution and integrated over all times. Clearly, this is not valid since the true two-point function interpolates between the IR and UV functions. Therefore, the 4-point function will not have a divergence at the h_0 mode, it will acquire a big value, but it will not diverge.

To see this explicitly, one can consider an example[13]:

$$G_{test}(\tau) = \left(\frac{1}{4\pi}\right)^{1/4} \frac{sgn(\tau)}{\sqrt{|J\tau|}} \frac{1}{\sqrt{1 + \frac{4}{|J\tau|}\left(\frac{1}{4\pi}\right)^{1/2}}} = \begin{cases} \left(\frac{1}{4\pi}\right)^{1/4} \frac{sgn(\tau)}{\sqrt{|J\tau|}} & , \quad J|\tau| \gg 1 \quad (IR) \\ \frac{1}{2}\, sgn(\tau) & , \quad J|\tau| \ll 1 \quad (UV) \end{cases} , \tag{29}$$

close to the IR, viz $J\tau| \gg 1$:

$$G_{test}(\tau) = \left(\frac{1}{4\pi}\right)^{1/4} \frac{sgn(\tau)}{\sqrt{|J\tau|}} \frac{1}{\sqrt{1 + \frac{4}{|J\tau|}\left(\frac{1}{4\pi}\right)^{1/2}}} \approx G_c(\tau) \left[1 - \frac{1}{|J\tau|\sqrt{\pi}} + \frac{3}{2\pi}\frac{1}{|J\tau|^2} + \cdots\right],$$

where $G_c(\tau) = G_{IR}(\tau)$. Thus, one can see that there is an additional contribution to the 2-point function to be consider in the kernel eigenvalues calculation, so that the divergence is removed. In the next section, we will give the thermal two-point function close to the conformal limit ($\beta J \gg 1$).

4 Thermodynamics in $q = 4$

Recalling the effective action from (10)

[13] This is an example of an interpolation function that converge to $G(\tau)$ in both limits, but it's not the real two-point function. To obtain the real one, we need to solve SD equations for a generic τ.

$$\frac{I_{eff}}{N} = -\frac{1}{2}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] + \frac{1}{2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{q}G(\tau_1,\tau_2)^q\right),$$

replacing SD equations above, gives us

$$\begin{aligned}
\frac{I_{eff}}{N} &= -\frac{1}{2}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] + \frac{1}{2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \left(\Sigma(\tau_1-\tau_2)G(\tau_1-\tau_2) - \frac{J^2}{q}G(\tau_1-\tau_2)^q\right) \\
&= -\frac{1}{2}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] + \frac{J^2}{2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \left(G(\tau_1-\tau_2)^q - \frac{1}{q}G(\tau_1-\tau_2)^q\right) \\
&= -\frac{1}{2}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] + \frac{J^2\beta}{2}\int_0^\beta d\tau_{12} \left(G(\tau_{12})^q - \frac{1}{q}G(\tau_{12})^q\right) \\
&= -\frac{1}{2}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] + \frac{(q-1)}{q}\frac{(\beta J)^2}{4\pi}\int_0^{2\pi} dv\; G\left(\frac{\beta v}{2\pi}\right)^q,
\end{aligned}$$

where $v = \frac{2\pi\tau_{12}}{\beta}$. Now, we aim to show that $G = G(\beta J)$. We can do this by dimensional analysis, since $G \sim \chi^2$, and from the kinetic term in the SYK Hamiltonian we know that the dimension in energy of the fermions is zero. Therefore, from the interacting term we can conclude that $[J] = 1$ (in energy). Hence, $[G] = 0$ and as the dimensionful parameters in energy available are β and J, the dependence of the 2-point function is on the dimensionless combination βJ, viz, $G = G(\beta J)$.

Therefore, the effective action depends only on the combination βJ and

$$\frac{\beta F}{N} = -\log Z = \frac{I_{eff}}{N} \equiv -f(\beta J).$$

Similarly, given that $E = -\partial_\beta\ logZ = -\partial_\beta\ f(\beta J) = -Jf'(\beta J)$, or (because f depends on the combination βJ)

$$\beta E = -\beta\partial_\beta f(\beta J) = -J\partial_J f(\beta J) = -J\beta f'(\beta J),$$

we can conclude that

$$\epsilon(\beta J) \equiv \frac{E}{J} = -f'(\beta J),$$

from which we can obtain an expansion for $\beta J \gg 1$.

Another way of calculating the energy is by the use of the action:

$$\beta E = \beta\partial_\beta\left(\frac{\beta F}{N}\right) = J\partial_J\left(\frac{I_{eff}}{N}\right) = -\frac{\beta J^2}{q}\int_0^\beta d\tau\ G(\tau)^q = \frac{\beta}{q}\ \partial_\tau G(\tau)|_{\tau\to 0^+},$$

where in the last equality, we replaced SD equations (use the KMS condition, the PH symmetry and take the corresponding limit). Obtaining,

$$\epsilon(\beta J) = \frac{1}{Jq}\ \partial_\tau G(\tau)|_{\tau\to 0^+}.$$

From the main reference (Maldacena and Stanford), we can write the free energy expansion in powers of $\beta J \gg 1$:

$$f(\beta J) = -(\beta J)\epsilon_0 + s_0 + \frac{2\pi^2\alpha_S}{\beta J} - \frac{2\pi^2\alpha_S\alpha_K}{(\beta J)^2|k'(2)|} + \cdots, \tag{30}$$

where $\epsilon_0 \approx -0.0406303$ is the ground state energy, $s_0 = 1/2 \log 2 - \int_0^\Delta dx\pi(1/2 - x)\tan(\pi x) \approx 0.232424$ is the zero-temperature entropy, $\alpha_S = \frac{\alpha_K}{128\pi} \approx 0.01001$, and $\alpha_K = -4k'(2)\alpha_0$ (with $k(h)$ as the eigenvalue function of the kernel).

From there, the energy expansion:

$$\epsilon(\beta J) = \epsilon_0 + \frac{c}{2}\frac{1}{(\beta J)^2} + c_2\frac{1}{(\beta J)^3} + c_p\frac{1}{(\beta J)^{h_1}} + \cdots, \tag{31}$$

where $c/2 = \frac{1}{48}\pi(2+3\pi)\alpha_0 \approx 0.198008$, $\alpha_0 \approx 0.2648056$ and $c_2 = \frac{\pi}{6}(2+3\pi)\alpha_0^2 \approx -0.41947$. Together with the exact relation $\frac{c_2}{(c/2)^2} = -\frac{384}{\pi(2+3\pi)} \approx -10.698763$.

Going back to the effective action, we can write

$$f(\beta J) = -\frac{I_{eff}}{N} = \frac{1}{2}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] - \frac{(q-1)}{q}\frac{\beta J^2}{2}\int_0^\beta d\tau'\ G\left(\tau'\right)^q,$$

using SD equations, as before,

$$f(\beta J) = \frac{1}{2}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] + \frac{(q-1)}{2}(\beta J)\epsilon(\beta J),$$

and since $\epsilon(\beta J) = -f'(\beta J)$, we can calculate the determinant[14]

$$\begin{aligned}\log\left[\det\left(\partial_\tau - \Sigma\right)\right] &= 2f(\beta J) + (q-1)(\beta J)f'(\beta J)\\ &= -(q+1)(\beta J)\epsilon_0 + 2s_0 - (q-3)\frac{c/2}{(\beta J)} - (q-2)\frac{c_2}{(\beta J)^2} + \cdots\end{aligned} \tag{32}$$

5 Correlation functions

As we know, the SYK model behaves as a CFT in the IR limit, therefore it shares some of its properties (the details on the derivation of the following formulae can be found in Appendix F). In particular, for any operator $\mathcal{O}_h$ with scaling dimension h, the n-point correlation functions, for $n \leq 3$, have the following form:

$$\begin{aligned}\langle \mathcal{O}_h(\tau)\rangle &= 0 && (1\ pt\ function)\\ \langle \mathcal{O}_h(\tau_1)\mathcal{O}_h(\tau_2)\rangle &= \frac{1}{|\tau_{12}|^{2h}} && (2\ pt\ function)\\ \langle \mathcal{O}_{h_1}(\tau_1)\mathcal{O}_{h_2}(\tau_2)\mathcal{O}_{h_3}(\tau_3)\rangle &= \frac{C_{h_1h_2h_3}}{|\tau_{12}|^{h_1+h_2-h_3}|\tau_{13}|^{h_1+h_3-h_2}|\tau_{23}|^{h_2+h_3-h_1}} && (3\ pt\ function),\end{aligned} \tag{33}$$

where $C_{h_1h_2h_3}$ is uniquely fixed.

For $n \geq 4$, the correlation functions become more complicated to calculate since we have to include the invariant cross-ratios. And, as such, we cannot fix their corresponding general form only using the covariance under the conformal group, as in the previous cases.

With this picture in mind, the next question to address is: What is the form of the primary operators $\mathcal{O}_h$ in SYK? Going back to the effective action, one can see that there is a $O(N)$ symmetry in the bilocal fields. This is easier to see if we write the correlation function as a vector product, namely, $G \sim \chi^T\chi$ and

[14]Note here that, at the end of the day, we have to regularize the determinant result since it diverges (UV divergence).

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_N \end{pmatrix}.$$

Since the operator dimensions are $h_m = 1 + 2m + 2\Delta + \epsilon_m$ (for $m = 1, 2, \ldots$ and $h_0 = 2$), for large m: $h_m = 1 + 2m + 2\Delta$. Then naively

$$\mathcal{O}_{h_m} = \chi_i \partial_\tau^{2m+1} \chi_i.$$

However, the real form of the primaries is (Gross and Rosenhaus: Generalization of SYK):

$$\mathcal{O}_{h_m} = \sum_{i=1}^{N} \sum_{k=0}^{2m+1} d_{mk} \partial_\tau^k \chi_i(\tau) \partial_\tau^{2m+1-k} \chi_i(\tau), \tag{34}$$

where the coefficients d_{mk} are chosen in such a way that the operators are primary.

[15]Since there is an infinite number of scaling dimensions, defined from the equality $k(h) = 1$, we have an infinite set of bilinear operators given by (34). Phenomenologically, the SYK model can be seen as a CFT plus a perturbation given by the infinite set of irrelevant primaries[16]:

$$I_{SYK} = I_{CFT} + \sum_h g_h \int d\tau \mathcal{O}_h(\tau), \tag{35}$$

where the sum runs over all the dimensions $h_0,\ h_1,\ h_2, \ldots$ and g_h is the perturbative coupling. From this expression, we can obtain the two-point function:

$$\begin{aligned} G(\tau_{12}) &= \frac{1}{Z} \int \mathcal{D}\chi_i \frac{1}{N} \chi_i(\tau_1)\chi_i(\tau_2) e^{-I_{SYK}} \\ &= \frac{1}{Z} \int \mathcal{D}\chi_i \frac{1}{N} \chi_i(\tau_1)\chi_i(\tau_2) e^{-I_{CFT}} e^{-\sum_h g_h \int d\tau \mathcal{O}_h(\tau)} \\ &= \frac{1}{Z} \int \mathcal{D}\chi_i \frac{1}{N} \chi_i(\tau_1)\chi_i(\tau_2) e^{-I_{CFT}} - \sum_h g_h \int d\tau_3 \frac{1}{Z} \int \mathcal{D}\chi_i \frac{1}{N} \chi_i(\tau_1)\chi_i(\tau_2) \mathcal{O}_h(\tau_3) e^{-I_{CFT}} + \cdots \\ &= G_c(\tau_{12}) - \sum_h g_h \int d\tau_3 \frac{1}{N} \langle \chi_i(\tau_1)\chi_i(\tau_2)\mathcal{O}_h(\tau_3) \rangle + \\ &\qquad + \frac{1}{2} \sum_{h_1,h_2} g_{h_1} g_{h_2} \int d\tau_3 \int d\tau_4 \frac{1}{N} \langle \chi_i(\tau_1)\chi_i(\tau_2)\mathcal{O}_{h_1}(\tau_3)\mathcal{O}_{h_2}(\tau_4) \rangle + \cdots \end{aligned} \tag{36}$$

To fix the functional form of the three-point function appearing in the first order perturbation of G, we use the conformal form shown in (33) together with the fact that such function must be chosen to solve the eigenvalue equation in the IR (neglecting the bare term in Figure 7):

$$v_h(\tau_1, \tau_2, \tau_0) = \int d\tau_3 \int d\tau_4 K(\tau_1, \tau_2, \tau_3, \tau_4) v_h(\tau_3, \tau_4, \tau_0),$$

[15]This part is taken from Tarnopolsky, et al: Excitation spectra...

[16]Such operators are irrelevant in the sense that as we approach to the IR, they give us the same CFT as if they were not there. That is at large τJ, we approach to the same CFT results and not have a new IR fixed point.

where $K(\tau_1, \tau_2, \tau_3, \tau_4)$ is the same kernel specified in the four-point function. But we know that

$$\int d\tau_3 \int d\tau_4 K(\tau_1, \tau_2, \tau_3, \tau_4) v_h(\tau_3, \tau_4, \tau_0) = k(h) v_h(\tau_1, \tau_2, \tau_0),$$

therefore we are looking for the solutions such that the eigenvalue of the kernel is 1 (which determine the scaling dimensions). Hence,

$$v_h(\tau_1, \tau_2, \tau_3) \equiv \frac{1}{N} \langle \chi_i(\tau_1)\chi_i(\tau_2)\mathcal{O}_h(\tau_3)\rangle = \frac{b^\Delta c_h sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta-h}|J\tau_{13}|^h|J\tau_{23}|^h}, \tag{37}$$

where c_h is determined by solving the eigenvalue equation and is

$$c_h^2 = \frac{1}{(q-1)b} \frac{(h-1/2)}{\pi \tan\frac{\pi h}{2}} \frac{\Gamma(h)^2}{\Gamma(2h)} \frac{1}{k'(h)}. \tag{38}$$

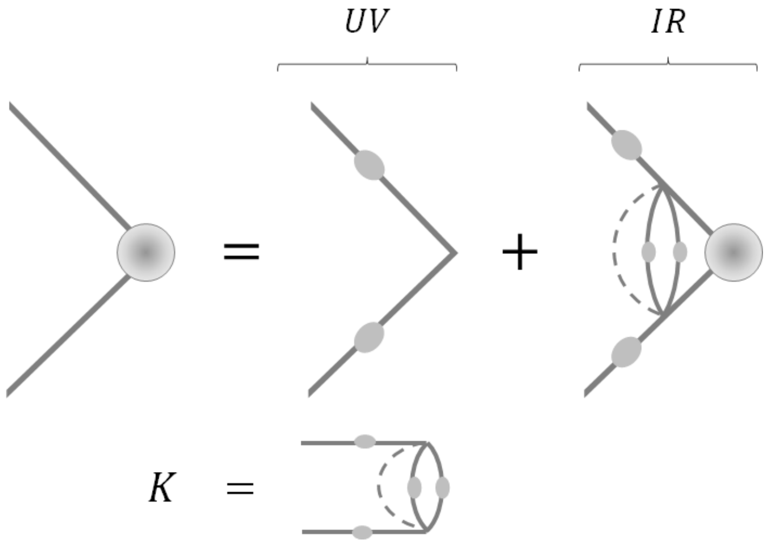


Figure 7: Diagrams that represent the 3-point function $\langle \chi_i(\tau_1)\chi(\tau_2)\mathcal{O}_h(\tau_3)\rangle$. The sum contains the same kernel K as in the four-point function case discussed before. The bare (tree level) diagram represents the UV limit, while the second term represents the IR one.

At zero temperature, up to third order correction[17] to $G(\tau)$ is

$$G(\tau) = G_c(\tau) \left[1 - \sum_m \frac{\alpha_m}{|J\tau|^{h_m-1}} - \sum_{m,n} \frac{a_{mn}\alpha_m\alpha_n}{|J\tau|^{h_m+h_n-2}} - \sum_{m,n,p} \frac{a_{mnp}\alpha_m\alpha_n\alpha_p}{|J\tau|^{h_m+h_n+h_p-3}} - \cdots \right], \tag{39}$$

with

$$\alpha_h^2 = \frac{g_h^2}{J^2} \frac{1}{(q-1)b\; k'(h)} \frac{\pi \tan\frac{\pi h}{2}}{(h-1/2)} \frac{\Gamma(2h)}{\Gamma(h)^2}.$$

So, the first terms in the corrected two-point correlation function read:

$$G(\tau) = G_c(\tau) \left[1 - \frac{\alpha_0}{|J\tau|} - \frac{a_{00}\alpha_0^2}{|J\tau|^2} - \frac{\alpha_1}{|J\tau|^{h_1-1}} - \frac{a_{000}\alpha_0^3}{|J\tau|^3} - \frac{a_{01}\alpha_0\alpha_1}{|J\tau|^{h_1}} - \cdots \right], \tag{40}$$

[17] The detailed calculation for the first order correction can be found in Appendix G, the other orders are referred to Tarnopolsky, et al.

where $h_1 \approx 3.7735$ and

$$a_{00} = \frac{(2\Delta+1)(2-2\Delta-\cos 2\pi\Delta)}{8\Delta\cos^2\pi\Delta} \qquad a_{000} = \frac{(\Delta+1)(2\Delta+1)(6\Delta-8+\cos 2\pi\Delta)}{24\Delta^2\cos^2\pi\Delta}.$$

To generalize the previous result to non-zero temperature (but still focusing in the strong coupling or low temperature regime, $\beta J \gg 1$), we can use the reparametrization symmetry[18]. So that, the expansion of $G(\tau)$, for $\tau \in [0,\beta]$, has the following form:

$$G_\beta(\tau) = G_c^\beta(\tau)\left[1 - \frac{\alpha_0}{\beta J} f_{h_0}(\tau) - \frac{a_{00}\,\alpha_0^2}{(\beta J)^2} f_{h_0,h_0}(\tau) - \frac{\alpha_1}{(\beta J)^{h_1-1}} f_{h_1}(\tau) - \frac{a_{000}\alpha_0^3}{(\beta J)^3} f_{h_0,h_0,h_0}(\tau) - \cdots\right], \tag{41}$$

where $a_{00} = \frac{9}{4}$, $a_{000} = -\frac{65}{4}$ and

$$G_c^\beta(\tau) = \left(\frac{\pi}{4}\right)^{1/4} \frac{1}{\sqrt{\beta J \sin\frac{\pi\tau}{\beta}}}, \qquad f_2(\tau) = 2 + \frac{\pi - \frac{\pi\tau}{\beta}}{\tan\frac{\pi\tau}{\beta}}, \qquad f_{2,2}(\tau) = \frac{\pi^2}{2}\left(\frac{2}{\sin^2\frac{\pi\tau}{\beta}} - 1\right).$$

In general, $f_h(\tau_{12})$ is obtained by solving

$$\delta^{(1)} G_\beta = -\sum_h g_h \int_0^\beta d\tau_3 \frac{1}{N}\left\langle \chi_i(\tau_1)\chi_i(\tau_2)\mathcal{O}_h(\tau_3)\right\rangle_\beta$$

in order to match (39), we obtain

$$\delta^{(1)} G_\beta = -G_c^\beta(\tau_{12}) \sum_m \frac{\alpha_m}{(\beta J)^{h_m-1}} f_{h_m}(\tau_{12}),$$

with

$$f_h(\tau_{12}) = A\int_0^\beta d\tau_3 \frac{\left|\sin\frac{\pi\tau_{12}}{\beta}\right|^h}{\left|\sin\frac{\pi\tau_{13}}{\beta}\right|^h \left|\sin\frac{\pi\tau_{23}}{\beta}\right|^h}$$

and A can be fixed by taking the zero temperature limit to math (39), namely

$$f_h(\tau) \underset{\beta\to\infty}{\longrightarrow} \left|\frac{\beta}{\tau}\right|^{h-1}.$$

The expressions for $f_{h,h'}(\tau)$ and $f_{h,h',h''}(\tau)$ are unknown and we can fix them via numerical calculations.

Another important quantity that can be obtained from the perturbed action (35) is the free energy that, up to second order, is

[18] This is possible because the expectation values are taken with respect to the CFT action, which has the reparametrization symmetry.

$$
\begin{aligned}
\beta\mathcal{F} &= -\log\left[\int \mathcal{D}\chi e^{-I_{SYK}}\right] \\
&= -\log\left[\int \mathcal{D}\chi e^{-I_{CFT}} e^{-\sum_h g_h \int d\tau \mathcal{O}_h(\tau)}\right] \\
&= -\log\left[\int \mathcal{D}\chi e^{-I_{CFT}}\left(1-\sum_h g_h \int_0^\beta d\tau \mathcal{O}_h(\tau) + \frac{1}{2}\sum_{h,h'} g_h g_{h'} \int_0^\beta d\tau_1 d\tau_2 \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\right)\right] \\
&= \beta\mathcal{F}_{CFT} - \log\left[1-\sum_h g_h \int_0^\beta d\tau \left\langle \mathcal{O}_h(\tau)\right\rangle_\beta + \frac{1}{2}\sum_{h,h'} g_h g_{h'} \int_0^\beta d\tau_1 d\tau_2 \left\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\right\rangle_\beta\right] \\
&= \beta\mathcal{F}_{CFT} + \sum_h g_h \int_0^\beta d\tau \left\langle \mathcal{O}_h(\tau)\right\rangle_\beta - \\
&\quad - \frac{1}{2}\sum_{h,h'} g_h g_{h'} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \left(\left\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\right\rangle_\beta - \left\langle \mathcal{O}_h(\tau_1)\right\rangle_\beta \left\langle \mathcal{O}_{h'}(\tau_2)\right\rangle_\beta\right),
\end{aligned}
$$

where $\beta\mathcal{F}_{CFT}/N = \beta E_0 - s_0$, s_0 is the zero-temperature entropy and E_0 the bare ground energy. The expectation values are with respect to the CFT action and in the last step it is important to notice that $\langle\mathcal{O}\rangle^2 \neq \langle\mathcal{O}\mathcal{O}\rangle$. In fact, the term $\left\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\right\rangle_\beta - \left\langle \mathcal{O}_h(\tau_1)\right\rangle_\beta \left\langle \mathcal{O}_{h'}(\tau_2)\right\rangle_\beta = \left\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\right\rangle_\beta^{con}$ represents the connected two point function of the operators $\mathcal{O}$. One can check that the higher order terms in g_h have only connected diagrams as well. An easy way to do this is to Taylor expand (up to fourth order) the function $\log(1-y)$, where

$$
\begin{aligned}
y = &\sum_h g_h \int_0^\beta d\tau \left\langle \mathcal{O}_h(\tau)\right\rangle_\beta - \frac{1}{2}\sum_{h,h'} g_h g_{h'} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \left\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\right\rangle_\beta + \\
&+ \frac{1}{3!}\sum_{h,h',h''} g_h g_{h'} g_{h''} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3 \left\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\mathcal{O}_{h''}(\tau_3)\right\rangle_\beta - \\
&- \frac{1}{4!}\sum_{h,h',h'',h'''} g_h g_{h'} g_{h''} g_{h'''} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3 \int_0^\beta d\tau_4 \left\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\mathcal{O}_{h''}(\tau_3)\mathcal{O}_{h'''}(\tau_4)\right\rangle_\beta
\end{aligned}
$$

So,

$$
\log(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4},
$$

and therefore, order by order in g_h, the terms are:

$O(1)$: $-\sum_h g_h \int_0^\beta d\tau \left\langle \mathcal{O}_h(\tau)\right\rangle_\beta$.

$O(2)$: $\frac{1}{2}\sum_{h,h'} g_h g_{h'} \int_0^\beta d\tau_1 d\tau_2 \left(\left\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\right\rangle_\beta - \left\langle \mathcal{O}_h(\tau_1)\right\rangle_\beta \left\langle \mathcal{O}_{h'}(\tau_2)\right\rangle_\beta\right)$.

$O(3)$:
$$
\begin{aligned}
&-\frac{1}{3!}\sum_{h,h',h''} g_h g_{h'} g_{h''} \int_0^\beta d\tau_1 d\tau_2 d\tau_3 \Big(\left\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\mathcal{O}_{h''}(\tau_3)\right\rangle_\beta - \\
&-3\left\langle \mathcal{O}_h(\tau_1)\right\rangle_\beta \left\langle \mathcal{O}_{h'}(\tau_2)\mathcal{O}_{h''}(\tau_3)\right\rangle_\beta^{con} - \left\langle \mathcal{O}_h(\tau_1)\right\rangle_\beta \left\langle \mathcal{O}_{h'}(\tau_2)\right\rangle_\beta \left\langle \mathcal{O}_{h''}(\tau_3)\right\rangle_\beta\Big).
\end{aligned}
$$

$$O(4): \quad -\frac{1}{4!}\sum_{h,h',h'',h'''} g_h g_{h'} g_{h''} g_{h'''}\int_0^\beta d\tau_1 d\tau_2 d\tau_3 d\tau_4 \Big(\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\mathcal{O}_{h''}(\tau_3)\mathcal{O}_{h'''}(\tau_4)\rangle_\beta -$$
$$-3\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\rangle_\beta^{con}\langle \mathcal{O}_{h''}(\tau_3)\mathcal{O}_{h'''}(\tau_4)\rangle_\beta^{con} - 6\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\rangle_\beta^{con}\langle \mathcal{O}_{h''}(\tau_3)\rangle_\beta \langle \mathcal{O}_{h'''}(\tau_4)\rangle_\beta$$
$$-4\langle \mathcal{O}_h(\tau_1)\rangle_\beta \langle \mathcal{O}_{h'}(\tau_2)\mathcal{O}_{h''}(\tau_3)\mathcal{O}_{h'''}(\tau_4)\rangle_\beta^{con} - \langle \mathcal{O}_h(\tau_1)\rangle_\beta \langle \mathcal{O}_{h'}(\tau_2)\rangle_\beta \langle \mathcal{O}_{h''}(\tau_3)\rangle_\beta \langle \mathcal{O}_{h'''}(\tau_4)\rangle_\beta \Big).$$

where $\langle \mathcal{O}_h(\tau)\rangle_\beta = \langle \mathcal{O}_h(\tau)\rangle_\beta^{con}$

Whence, the terms entering in the expansion of the free energy are only connected correlation functions. So, from now on we will remove the superscript $\langle\,\rangle^{con}$, but keeping in mind that those expectation values are represented by connected diagrams. So, the free energy reads

$$\beta\mathcal{F} = N\beta E_0 - Ns_0 + \sum_h g_h \int_0^\beta d\tau \,\langle \mathcal{O}_h(\tau)\rangle_\beta - \frac{1}{2}\sum_{h,h'} g_h g_{h'}\int_0^\beta d\tau_1 d\tau_2\,\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\rangle_\beta +$$
$$+\frac{1}{3!}\sum_{h,h',h''} g_h g_{h'} g_{h''}\int_0^\beta d\tau_1 d\tau_2 d\tau_3\,\langle \mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\mathcal{O}_{h''}(\tau_3)\rangle_\beta + \ldots$$

It has been argued that one-point functions in thermal CFT are not necessarily zero and have the constant value:

$$\langle \mathcal{O}_h\rangle_\beta = \frac{N b_h}{(\beta J)^h},$$

where the coefficients b_h are fixed by the conformal two point function as follows (for $\tau \to 0$):

$$\begin{aligned} G_c^\beta(\tau) &= \frac{b^\Delta sgn(\tau)}{\left|\frac{\beta J}{\pi}\sin\frac{\pi\tau}{\beta}\right|^{2\Delta}} \\ &\underset{\tau\to 0}{\to} \frac{b^\Delta sgn(\tau)}{\left|\frac{\beta J}{\pi}\left[\frac{\pi\tau}{\beta} - \frac{1}{3!}\left(\frac{\pi\tau}{\beta}\right)^3 + \frac{1}{5!}\left(\frac{\pi\tau}{\beta}\right)^5 + \cdots\right]\right|^{2\Delta}} \\ &= \frac{b^\Delta sgn(\tau)}{|J\tau|^{2\Delta}\left|1 - \frac{1}{3!}\left(\frac{\pi\tau}{\beta}\right)^2 + \frac{1}{5!}\left(\frac{\pi\tau}{\beta}\right)^4 + \cdots\right|^{2\Delta}} \\ &= \frac{b^\Delta sgn(\tau)}{|J\tau|^{2\Delta}}\left(1 + \frac{\pi^2}{3}\Delta\left|\frac{\tau}{\beta}\right|^2 + \frac{\pi^4}{90}\Delta(5\Delta+1)\left|\frac{\tau}{\beta}\right|^4 + \ldots\right). \end{aligned} \tag{42}$$

On the other hand, using the OPE expansion[19] of $G(\tau)$, gives us:

$$G_\beta(\tau) = \frac{b^\Delta sgn(\tau)}{|J\tau|^{2\Delta}}\left(1 + \sum_h c_h |J\tau|^h \langle \mathcal{O}_h\rangle_\beta\right), \tag{43}$$

[19]The OPE expansion allows us to write

$$G_\beta(\tau) = \sum_H C_H(\tau)\langle \mathcal{O}_H\rangle_\beta = \sum_H \frac{b^\Delta sgn(\tau)}{|J\tau|^{2\Delta-H}} c_H \langle \mathcal{O}_H\rangle_\beta = \frac{b^\Delta sgn(\tau)}{|J\tau|^{2\Delta}} + \sum_h \frac{b^\Delta sgn(\tau)}{|J\tau|^{2\Delta-h}} c_h \langle \mathcal{O}_h\rangle_\beta,$$

where in the last equality we separated the identity term ($H=0$, $c_0=1$, and $\mathcal{O}_0 = \mathbb{I}$) from the anomalous dimensions terms $H=h$. See appendix H for more details on the OPE expansion.

comparing both expressions, we can fix b_h. For example,

$$N b_{h_0} c_{h_0} = \frac{\pi^2}{3}\Delta, \qquad N b_{h=4} c_{h=4} = \frac{\pi^4}{90}\Delta(5\Delta+1), \cdots$$

An important note here is that $b_h = 0$ for $h \neq 2k,\ k \in \mathbb{Z}$. So, from CFT arguments the one-point functions

$$\langle \mathcal{O}_{h_m} \rangle_\beta = 0,\ m = 1, 2, 3, \ldots$$

Therefore, the free energy gets the form:

$$\beta\mathcal{F} = N\beta E_0 - N s_0 + \beta N \sum_h \frac{g_h b_h}{(\beta J)^h} - \frac{1}{2}\sum_h g_h^2 \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \, \langle \mathcal{O}_h(\tau_1)\mathcal{O}_h(\tau_2)\rangle_\beta + \cdots$$

where the third term has only contributions from $h = 2k,\ k \in \mathbb{N}$ (according to the previous analysis), the rest are zero: $b_h = 0$.

The fourth term can be calculated explicitly by introducing a UV cutoff[20] $\varepsilon \sim 1/J$:

$$\begin{aligned} g_h^2 \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \, \langle \mathcal{O}_h(\tau_1)\mathcal{O}_h(\tau_2)\rangle_\beta &= N g_h^2 \beta \int_\varepsilon^{\beta-\varepsilon} d\tau \left(\frac{\pi}{\beta J \sin\frac{\pi\tau}{\beta}}\right)^{2h} \\ &= \frac{N(g_h^2/J^2)(\beta J)}{(h-1/2)(\varepsilon J)^{2h-1}} + \frac{N(g_h^2/J^2)}{(\beta J)^{2h-2}} \frac{\pi^{2h-1/2}\Gamma(\frac{1}{2}-h)}{\Gamma(1-h)}, \end{aligned}$$

where the first term represents a correction to the ground energy. Finally, we arrive at

$$\beta\mathcal{F} = \beta\mathcal{F}_{CFT} + N\sum_h \frac{(g_h/J)b_h}{(\beta J)^{h-1}} - \frac{1}{2}\sum_h \left[\frac{N(g_h^2/J^2)(\beta J)}{(h-1/2)(\varepsilon J)^{2h-1}} + \frac{N(g_h^2/J^2)}{(\beta J)^{2h-2}} \frac{\pi^{2h-1/2}\Gamma(\frac{1}{2}-h)}{\Gamma(1-h)}\right] + \cdots$$

And using the thermodynamic relation $\epsilon = E/J = -\partial_\beta \log Z/J = \partial_\beta(\beta\mathcal{F}/(JN))$, the expression for the energy is:

$$\begin{aligned} \epsilon = \epsilon_0 &- \sum_h \frac{(g_h/J) b_h (h-1)}{(\beta J)^h} + \\ &- \frac{1}{2}\sum_h \left[\frac{(g_h^2/J^2)}{(h-1/2)(\varepsilon J)^{2h-1}} - \frac{(2h-2)(g_h^2/J^2)}{(\beta J)^{2h-1}} \frac{\pi^{2h-1/2}\Gamma(\frac{1}{2}-h)}{\Gamma(1-h)}\right] + \cdots \end{aligned} \tag{44}$$

Or equivalently,

$$\epsilon = \epsilon_0 + \frac{c_2}{(\beta J)^2} + \frac{c_3}{(\beta J)^3} + \frac{c_4}{(\beta J)^4} + \frac{c_5}{(\beta J)^5} + \frac{c_6}{(\beta J)^6} + \frac{c_{2h_1-1}}{(\beta J)^{2h_1-1}} + \frac{c_7}{(\beta J)^7} + \ldots \tag{45}$$

with $h_1 \approx 3.773535$, $\epsilon_0 \approx -0.0406303$, $c_2 = \frac{1}{48}\pi(2+3\pi)\alpha_0 \approx 0.198008$, $c_3 = \frac{1}{6}\pi(2+3\pi)\alpha_0^2 \approx -0.41947$, and $\frac{c_3}{(c_2)^2} = -\frac{384}{\pi(2+3\pi)} \approx -10.698763$. Notice that the first non-integer exponent comes from the two point function in the free energy calculation $\langle \mathcal{O}_{h_1}\mathcal{O}_{h_1}\rangle$ [21]. Also, numerically, we can see that terms $1/(\beta J)^p$ with $p = h_1,\ h_1+1,\ h_1+2$ or h_2 are absent. Therefore, it is natural to assume that we do not expect terms with powers $p = h_m + n$, with $n = 0, 1, 2, \ldots$ to be present in the energy expansion.

[20] This cutoff is of order $1/J$ because at order $\beta J \sim 1$ the conformal limit stops working.

[21] Note that if any of the terms $\frac{1}{(\beta J)^{h_m}}$ existed, it would imply non-zero one point functions $\langle \mathcal{O}_{h_m}\rangle$.

Note. *It is important to take into account that this (phenomenological) treatment is not completely clear yet, specially for the h_0 case, since there are some caveats. Although, starting from* (35) *gives a natural way of obtaining the observables, one has to be careful in trusting such calculations. THAT IS WHY IT IS BETTER TO USE NUMERICS TO SOLVE THE MODEL AND CALCULATE BOTH THE ENERGY AND THE 2-POINT CORRELATION FUNCTION EXPANSIONS.*

It is needed very high precision in the numerical results in order to test the different predictions of the SYK model in a trustworthy way.

6 Chaos

[22]Let us first talk about semi-classical chaos: In a classical system, an easy probe of chaos is the high dependence of the trajectories on the initial conditions (butterfly effect). A chaotic system has nearby trajectories that diverge exponentially fast in time, viz.

$$\frac{\partial q(t)}{\partial q(0)} = \{q(t), p(0)\}_{PB} \sim e^{\lambda_L t},$$

where λ_L is called the Lyapunov exponent and $t > 0$.

Considering a semi-classical quantum system ($\hbar \ll 1$), we can approximate the Poisson bracket with the commutator, i.e. $\{\ \} \to \frac{1}{i\hbar}[\ ]$. However, there are some caveats in this limit: 1. Since there can be phase cancellations, we need to square the commutator and 2. We need to take the expectation value in some state. A general choice for this is the (canonical ensemble) thermal state, leading us to consider the quantity:

$$C(t) = -\left\langle [W(t), V(0)]^2 \right\rangle_\beta ,$$

where as before $\langle\ \rangle_\beta = \mathrm{Tr} e^{-\beta H}/Z$. Writing $C(t)$ explicitly and using the KMS condition[23] we have:

$$\begin{aligned} C(t) = &- \langle W(t)V(0)^2 W(t)\rangle_\beta - \langle V(0)W(t)^2 V(0)\rangle_\beta + \langle W(t)V(0)W(t)V(0)\rangle_\beta + \\ &+ \langle V(0)W(t)V(0)W(t)\rangle_\beta \\ = &- \langle W(0)W(i\beta)V(t+i\beta)^2\rangle_\beta - \langle V(-t-i\beta)V(-t)W(0)^2\rangle_\beta + \\ &+ \langle V(0)W(t)V(0)W(t)\rangle_\beta + \langle V(0)W(t)V(0)W(t+i\beta)\rangle_\beta , \end{aligned}$$

the first two correlators are in Lorentzian time order, but the last two are not. These are called Out-of-Time-Order (OTO) correlation functions. The typical behavior of the function $C(t)$ is as follows: after a time of order β, there is a region of Lyapunov growth, however when the commutator obtains macroscopic values (around Ehrenfest time $t_s \sim \frac{1}{\lambda_L} \log \frac{1}{\hbar}$), $C(t)$ starts saturating exponentially to its late time average (called the Ruelle region, macroscopic average value).

Now we can focus on studying the behavior of $C(t)$ in a theory holographically dual to Einstein gravity by studying certain shockwaves sent into an AdS-Schwartzschild black hole. In summary, the growth of the Lyapunov part is a consequence of the exponential redshift near the horizon. The result is that the four-point function behaves, at late times, as

[22]This section is mainly taken from Sarosi.

[23]Recall that $\langle A(t)B(0)\rangle_\beta = \langle B(0)A(t+i\beta)\rangle_\beta$, and for correlations of more than two operators, play with the cyclicity of the trace.

$$F_{VWVW} \sim \beta \frac{\Delta^2}{C} e^{\frac{2\pi}{\beta} t}.$$

Suggesting that we have a Lyapunov exponent of $\lambda_L = 2\pi/\beta$ (this value is also obtained in higher dimensional black holes and it is argued that this is the maximal Lyapunov exponent that a chaotic quantum system with a classical limit can have).

[24]Let us now turn to the SYK language and consider the OTO four-point function (for times longer that the dissipation time and shorter than the scrambling time $\lambda_L^{-1} \ll t \ll \lambda_L^{-1} \log N$):

$$\langle \chi_i(0)\chi_j(t)\chi_i(0)\chi_j(t)\rangle_\beta \sim \frac{1}{N} e^{\lambda_L t}.$$

Taking SD equation for the OTO 4-point function and plugging in the previous expression, it was found by Kitaev that $\lambda_L = 2\pi/\beta$ for $\beta J \gg 1$. That is, SYK saturates the maximally chaotic bound.

7 Applications

Section mainly based on the paper by Rosenhaus (An Introduction to the SYK model).

7.1 AdS/CFT

Since the full SYK model breaks conformal invariance, then it should not be thought of as dual to AdS$_2$. Instead, one should consider an AdS$_2$ embedded in a higher dimensional space (as a near-horizon limit of an extremal ($E = 0$) charged Reisnner-Nordström black hole in asymptotic AdS), while the dual of this bulk is not SYK, it may be that the IR limit is.

At low energies, SYK is dominated by the $h = 2$ mode, which is described by the Schwarzian (consequence of nearly conformal invariance). On the AdS$_2$ side, it is natural to consider Jackiw-Teitelboim (JT) dilaton gravity. Dilaton gravity theories arise from compactifying gravity in higher dimensions down to two with the dilaton playing the role of the size of the extra dimension. It has been shown that dilaton gravity in AdS$_2$ is equivalent to Schwarzian theory (as a consequence of the pattern of symmetry breaking).

With this in mind, in principle, we know the bulk in AdS$_2$ because we have solved the model in the IR. Then, the dictionary works as follows:

Boundary	**Bulk**
$\mathcal{O}_{h_n} \sim \frac{1}{N}\chi_i \partial_\tau^{1+2n}\chi_i$	$\varphi_n \equiv$ particle $m_n^2 = h_n(h_n - 1)$

where the masses are chosen in such a way that they match the two-point functions of the operators $\mathcal{O}_h$.

So, in the bulk we have a tower of fields of increasing masses. The number of fields is infinite and they are parametrized by n. However, the dual of the fermions χ_i is still an open question. The 'Bulk Theory' can be written as

$$\mathcal{L} = \sum_n (\partial\varphi_n)^2 + m_n^2\varphi_n^2 + \Lambda_{nkl}\varphi_n\varphi_k\varphi_l + \cdots$$

where the couplings are, in principle, known because we have solved SYK (and use the dictionary). Though, it is not clear where the set of numbers m_n, Λ_{nkl}, etc. come from, i.e. there is no presently a bulk description yet.

[24]This part is taken from Polchinski and Rosenhaus.

7.2 Strange metals

Considering a lattice of SYK models, for example

$$\mathcal{L} = \sum_x \sum_i c^\dagger_{i,x}(\partial_\tau - \mu)c_{i,x} - \sum_{\langle xx' \rangle} \sum_{i,j} t_{ij,xx'} c^\dagger_{i,x} c_{j,x'} - \sum_x \sum_{i,j,k,l} J_{ijkl,x} c^\dagger_{i,x} c^\dagger_{j,x} c_{k,x} c_{l,x},$$

where x represents the lattice site, $\langle\ \rangle$ sum over nearest neighbors, $t_{ij,xx'}$ are random hopping couplings, and $J_{ijkl,x}$ is the random interaction coupling per site x. Notice that now $c_{i,x}$ are complex fermions. This model exhibits features of a strongly correlated metal, with resistivity scaling linearly with T, at high temperatures, and Fermi liquid behavior at low temperatures.

Nevertheless, there are two main limitations here: neither all-to-all interactions nor large N are present in real metals, and this features are essential to solve the SYK model.

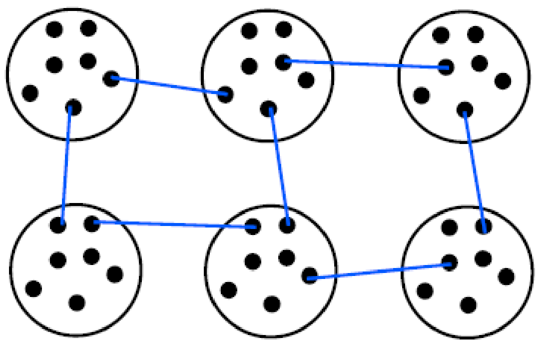


Figure 8: A cartoon representation of a SYK lattice of 'quantum dots' with quartic all-to-all interactions, image taken from Rosenhaus (An Introduction to the SYK model).

Note. *The time dependent green function determines the transport properties of a system (as the electrical resistance and complex dielectric constant as a function of the field frequency) as well as inelastic particle scattering processes in solids.*

A Path Integrals for fermions and Matsubara frequencies

[25] Let Ψ and $\Psi^\dagger$ be two fermionic (ladder) operators, namely

$$\left\{\Psi^\dagger, \Psi\right\} = 1, \qquad \left\{\Psi^\dagger, \Psi^\dagger\right\} = 0, \qquad \{\Psi, \Psi\} = 0$$

and the number operator $\mathcal{N} = \Psi^\dagger\Psi$. Since, $\mathcal{N}^2 = \mathcal{N}$, the possible eigenvalues for $\mathcal{N}$ are either 0 or 1. Therefore,

$$\Psi^\dagger |0\rangle = |1\rangle, \qquad \Psi |1\rangle = |0\rangle, \qquad \Psi^\dagger |1\rangle = 0, \qquad \Psi |0\rangle = 0.$$

Now, let's define the state,

$$|\psi\rangle = |0\rangle - \psi |1\rangle,$$

where ψ is a Grassmann number, this state is a fermion coherent state, because it is an eigenvector of all 'annihilation' operators, viz,

$$\begin{aligned}
\Psi |\psi\rangle &= \Psi |0\rangle - \Psi\psi |1\rangle \\
&= \psi\Psi |1\rangle \\
&= \psi |0\rangle \\
&= \psi (|0\rangle - \psi |1\rangle) \\
&= \psi |\psi\rangle .
\end{aligned}$$

Similarly, $\langle\bar\psi| \Psi^\dagger = \langle\bar\psi| \bar\psi$. Note that $\bar\psi$ is not the hermitian conjugate of ψ, we should treat them as independent Grassmann variables, hence $\langle\bar\psi| \neq (|\psi\rangle)^\dagger$.

Let's now enumerate some of the important results from Grassmann numbers before writing the path integral:

i) $\langle\bar\psi|\psi\rangle = 1 + \bar\psi\psi = e^{\bar\psi\psi}$.

ii) Any function of n Grassmann variables can be written as a polynomial with 2^n terms, i.e. $f(\psi_1, \psi_2, \ldots, \psi_n) = (a_1 + b_1\psi_1)(a_2 + b_2\psi_2)\cdots(a_n + b_n\psi_n)$.

iii) Since $\int d\bar\psi d\psi\ \bar\psi\psi = -1$, then $\int d\bar\psi d\psi\ e^{-a\bar\psi\psi} = a$. Therefore, considering ψ to be a column vector of any size, $\bar\psi$ a row vector of the same size and M a square matrix (of the size of ψ), $\int \left[d\bar\psi d\psi\right]\ e^{-\bar\psi M\psi} = \det M$.

iv) The completeness relation: $\int d\bar\psi d\psi\ \ |\psi\rangle \langle\bar\psi| e^{-\bar\psi\psi} = \mathbb{I}$.

v) $\langle-\bar\psi| = \langle 0| + \langle 1| \bar\psi = \langle 0| - \bar\psi \langle 1|$, so that $\langle-\bar\psi|\psi\rangle = 1 - \bar\psi\psi = e^{-\bar\psi\psi}$.

vi) $\int d\bar\psi d\psi\ \ \langle-\bar\psi| \Omega |\psi\rangle e^{-\bar\psi\psi} = Tr\ \Omega$.

Now, following the path integral approach to write the partition function, we have

$$Z = Tr\left(e^{-\beta H}\right),$$

where $H = H(\Psi^\dagger, \Psi)$ is the normal-ordered Hamiltonian. Plus, discretizing time $\epsilon = \beta/N$:

[25]This section is taken from Shankar (RG approach to interacting fermions)

$$
\begin{aligned}
\langle\bar{\psi}_{i+1}|\, e^{-\epsilon H}\, |\psi_i\rangle &\approx \langle\bar{\psi}_{i+1}|\,(1-\epsilon H(\Psi^\dagger,\Psi))\,|\psi_i\rangle \\
&\approx \langle\bar{\psi}_{i+1}|\,(1-\epsilon H(\bar{\psi}_{i+1},\psi_i))\,|\psi_i\rangle \\
&= \langle\bar{\psi}_{i+1}|\psi_i\rangle\, e^{-\epsilon H(\bar{\psi}_{i+1},\psi_i)} \\
&= e^{\bar{\psi}_{i+1}\psi_i} e^{-\epsilon H(\bar{\psi}_{i+1},\psi_i)}
\end{aligned}
$$

Thus, since $e^{-\beta H} = \lim_{N\to\infty} e^{-\epsilon H N}$, then introducing the completeness relation into every $e^{-\epsilon H}$ and letting $\bar{\psi}_N = -\bar{\psi}_1$, $\psi_N = -\psi_1$ (anti-symmetric boundary conditions[26]):

$$
\begin{aligned}
Z &= \int d\bar{\psi}_1 d\psi_1\ \langle-\bar{\psi}_1|\, e^{-\beta H}\, |\psi_1\rangle\, e^{-\bar{\psi}_1\psi_1} \\
&= \lim_{N\to\infty}\int \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i\, \langle\bar{\psi}_N|\, e^{-\epsilon H}\, |\psi_{N-1}\rangle\, e^{-\bar{\psi}_{N-1}\psi_{N-1}}\, \langle\bar{\psi}_{N-1}| \times \cdots \times |\bar{\psi}_2\rangle\, e^{-\bar{\psi}_2\psi_2}\, \langle\bar{\psi}_2|\, e^{-\epsilon H}\, |\psi_1\rangle\, e^{-\bar{\psi}_1\psi_1} \\
&= \lim_{N\to\infty}\int \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i\ e^{\bar{\psi}_N\psi_{N-1}} e^{-\epsilon H(\bar{\psi}_N,\psi_{N-1})} e^{-\bar{\psi}_{N-1}\psi_{N-1}} \cdots e^{-\bar{\psi}_2\psi_2} e^{\bar{\psi}_2\psi_1} e^{-\epsilon H(\bar{\psi}_2,\psi_1)} e^{-\bar{\psi}_1\psi_1} \\
&= \lim_{N\to\infty}\int \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i\ e^{\sum_{j=1}^{N-1}\left[\bar{\psi}_{j+1}\psi_j - \bar{\psi}_j\psi_j - \epsilon H(\bar{\psi}_{j+1},\psi_j)\right]} \\
&= \lim_{N\to\infty}\int \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i\ e^{\sum_{j=1}^{N-1}\epsilon\left[\frac{\left(\bar{\psi}_{j+1}-\bar{\psi}_j\right)}{\epsilon}\psi_j - H(\bar{\psi}_{j+1},\psi_j)\right]} \\
&= \int \left[d\bar{\psi}d\psi\right]\ e^{\int_0^\beta d\tau\ \left[\frac{\partial\bar{\psi}}{\partial\tau}\psi - H(\bar{\psi},\psi)\right]} \\
&= \int \left[d\bar{\psi}d\psi\right]\ e^{-\int_0^\beta d\tau\left[\bar{\psi}(\tau)\frac{\partial\psi(\tau)}{\partial\tau} + H(\bar{\psi}(\tau),\psi(\tau))\right]}.
\end{aligned}
$$

If we turn to Fourier space, by writing[27]

$$
\bar{\psi}(\tau) = \frac{1}{\beta}\sum_n e^{i\omega_n\tau}\tilde{\bar{\psi}}(\omega), \qquad \psi(\tau) = \frac{1}{\beta}\sum_n e^{-i\omega_n\tau}\tilde{\psi}(\omega)
$$

and apply the anti-symmetric boundary conditions:

$$
\psi(\beta) = -\psi(0), \qquad \bar{\psi}(\beta) = -\bar{\psi}(0)
$$

implies that

$$
e^{\pm i\omega_n\beta} = -1 \implies \omega_n = \frac{(\pi + 2\pi n)}{\beta},
$$

for $n \in \mathbb{Z}$, or

$$
\omega_n = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\right)
$$

which are called Matsubara frequencies.

[26]The first condition is obvious, the second one is for the Fourier transforms to be well defined.

[27]Notice that we have written the Fourier transforms as if they were complex conjugates, but take into account that they are not, this just ease the calculations.

B Free two-point function and exact series in Fourier modes

The free (zero-T) two-point function is defined as:

$$G_0(\tau) = \frac{1}{N}\sum_{i,j=1}^{N} G^0_{ij}(\tau),$$

where $G^0_{ij}(\tau) \equiv \langle \mathcal{T}\chi_i(\tau)\chi_j(0)\rangle$, $\chi_i(\tau) = e^{\tau H}\chi_i(0)e^{-\tau H}$, and H is the Hamiltonian of the theory, which in this case is zero (non-interacting case). Therefore, $\chi_i(\tau) = \chi_i(0) - \chi_i$. Thus,

$$\begin{aligned}
G^0_{ij}(\tau) &\equiv \langle \mathcal{T}\chi_i(\tau)\chi_j(0)\rangle \\
&= \Theta(\tau)\langle\chi_i\chi_j\rangle - \Theta(-\tau)\langle\chi_j\chi_i\rangle \\
&= \Theta(\tau)\langle\chi_i\chi_j\rangle - \Theta(-\tau)\left(\delta_{ij} - \langle\chi_i\chi_j\rangle\right) \\
&= \langle\chi_i\chi_j\rangle\left(\Theta(\tau)+\Theta(-\tau)\right) - \Theta(-\tau)\delta_{ij} \\
&= \langle\chi_i\chi_j\rangle - \Theta(-\tau)\delta_{ij}.
\end{aligned}$$

Leading to[28]

$$G^0_{ij}(\tau) = \begin{cases}\langle\chi_i\chi_j\rangle = -\langle\chi_j\chi_i\rangle, & i\neq j\\ \frac{1}{2} - \Theta(-\tau), & i=j\end{cases} = \begin{cases}0, & i\neq j\\ \frac{1}{2}\,sgn(\tau), & i=j\end{cases} = \frac{1}{2}\,sgn(\tau)\,\delta_{ij}$$

$$\Longrightarrow\; G_0(\tau) = \frac{1}{2}\,sgn(\tau).$$

B.1 Sums in Matsubara modes

There is a way of verifying what is the free non-zero Temperature correlator, using the discrete Fourier expansion in Matsubara frequencies. Of course, writing it in Fourier modes allows us to find $G_0(iw_n)$ by integrating $G_0(\tau)$. But, we want to calculate explicitly the sum over frequencies, and for this, let's start by doing some of these discrete Fourier sums to see how it works:

The function: $\tanh\left(\frac{\beta z}{2}\right) = \frac{e^{\frac{\beta z}{2}} - e^{-\frac{\beta z}{2}}}{e^{\frac{\beta z}{2}} + e^{-\frac{\beta z}{2}}}$ has poles when $e^{\frac{\beta z}{2}} = -e^{-\frac{\beta z}{2}}$, that is when $\beta z = i\pi + 2in\pi$, i.e

$$z = i\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right) = i\omega_n,$$

where $\omega_n = \frac{2\pi}{\beta}\left(n+\frac{1}{2}\right)$ are the Matsubara frequencies for fermions.

Thus, expanding $\tanh\left(\frac{\beta z}{2}\right)$ around $i\omega_n$ gives,

[28] Noticing that the $i\neq j$ part is independent of τ, and using $G(\tau) = -G(-\tau)$. And that $\chi_i\chi_i = \frac{1}{2}$ from the anti-commutation relations.

$$
\begin{aligned}
\tanh\left(\frac{\beta z}{2}\right) &= \frac{\sinh\left(\frac{\beta z}{2}\right)}{\cosh\left(\frac{\beta z}{2}\right)} \\
&= \frac{\sinh(i\beta\omega_n/2) + \frac{\beta}{2}\cosh(i\beta\omega_n/2)(z-i\omega_n) + \left(\frac{\beta}{2}\right)^2 \sinh(i\beta\omega_n/2)\frac{(z-i\omega_n)^2}{2!} + \dots}{\cosh(i\beta\omega_n/2) + \frac{\beta}{2}\sinh(i\beta\omega_n/2)(z-i\omega_n) + \left(\frac{\beta}{2}\right)^2 \cosh(i\beta\omega_n/2)\frac{(z-i\omega_n)^2}{2!} + \dots} \\
&= \frac{1 + \left(\frac{\beta}{2}\right)^2 \frac{(z-i\omega_n)^2}{2!} + \left(\frac{\beta}{2}\right)^4 \frac{(z-i\omega_n)^4}{4!} + \dots}{\left(\frac{\beta}{2}\right)(z-i\omega_n) + \left(\frac{\beta}{2}\right)^3 \frac{(z-i\omega_n)^3}{3!} + \left(\frac{\beta}{2}\right)^5 \frac{(z-i\omega_n)^5}{5!} + \dots} \\
&= \frac{1 + \left(\frac{\beta}{2}\right)^2 \frac{(z-i\omega_n)^2}{2!} + \left(\frac{\beta}{2}\right)^4 \frac{(z-i\omega_n)^4}{4!} + \dots}{\left(\frac{\beta}{2}\right)(z-i\omega_n)\left[1 + \left(\frac{\beta}{2}\right)^2 \frac{(z-i\omega_n)^2}{3!} + \left(\frac{\beta}{2}\right)^4 \frac{(z-i\omega_n)^4}{5!} + \dots\right]}
\end{aligned}
$$

Therefore, the residue is:

$$
\operatorname*{Res}_{z=i\omega_n} \tanh\left(\frac{\beta z}{2}\right) = \frac{2}{\beta},
$$

and by the residue theorem, we have

$$
\sum_n f(i\omega_n) = \frac{\beta}{4\pi i}\oint_C dz \tanh\left(\frac{\beta z}{2}\right) f(z),
$$

where C is the contour along the imaginary axis enclosing the poles. The next step to calculate the sum is to shift the contour so that the new C' encloses the poles of $f(z)$ (we can always do this because the result is independent of the contour we choose).

To continue this analysis, and since $\tanh\left(\frac{\beta z}{2}\right) = \pm\frac{\left(e^{\pm\beta z}-1\right)}{e^{\pm\beta z}+1}$, it is easier to redo the previous calculations for the function $\frac{1}{1+e^{\pm\beta z}}$. It has poles when $e^{\pm\beta z} = -1$, that is $\pm\beta z = i\pi + 2\pi i n$, or

$$
z = \pm i\omega_n
$$

So,

$$
\begin{aligned}
\frac{1}{1+e^{\pm\beta z}} &= \frac{1}{1 + (-1) \pm (-1)(z-i\omega_n)\beta + (-1)\frac{(z-i\omega_n)^2}{2!}\beta^2 + \dots} \\
&= \frac{1}{\mp(z-i\omega_n)\beta\left[1 \pm \frac{(z-i\omega_n)^2}{2!}\beta + \frac{(z-i\omega_n)^3}{3!}\beta^2 \dots\right]} \\
\implies \operatorname*{Res}_{z=i\omega_n} \frac{1}{1+e^{\pm\beta z}} &= \mp\frac{1}{\beta}.
\end{aligned}
$$

Hence

$$
\sum_n f(i\omega_n) = \frac{\beta}{2\pi i}\oint_C dz \frac{f(z)}{1+e^{-\beta z}} = -\frac{\beta}{2\pi i}\oint_C dz \frac{f(z)}{1+e^{\beta z}},
$$

we can choose which one to use, depending on $f(z)$, the key part is that when we shift the contour to the poles of f, the integral over the infinite part of it vanishes.

B.1.1 Examples

The sum (usually encountered in Matsubara Fourier transforms)

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{-i\omega_n\tau}}{\omega-i\omega_n},$$

with ω constant and $\tau \in [0,\beta]$ can be obtained by applying the previous complex calculation, viz

$$\begin{aligned}
\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{-i\omega_n\tau}}{\omega-i\omega_n} &= \frac{1}{2\pi i}\oint_C dz\frac{1}{(1+e^{-\beta z})}\frac{e^{-z\tau}}{(\omega-z)}\\
&= \frac{1}{2\pi i}\oint_{C'} dz\frac{1}{(1+e^{-\beta z})}\frac{e^{-z\tau}}{(\omega-z)}\\
&= \frac{2\pi i}{2\pi i}\operatorname*{Res}_{z=\omega}\frac{e^{-z\tau}}{(1+e^{-\beta z})(\omega-z)}\\
&= \frac{e^{-\omega\tau}}{1+e^{-\beta\omega}}\\
&= \frac{e^{(\beta-\tau)\omega}}{1+e^{\beta\omega}}
\end{aligned}$$

where in the second line, we shifted the contour C to C' so that we enclose the pole at $z=\omega$.

Similarly,

$$\begin{aligned}
\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{i\omega_n\tau}}{\omega-i\omega_n} &= -\frac{1}{2\pi i}\oint_C dz\frac{1}{(1+e^{\beta z})}\frac{e^{z\tau}}{(\omega-z)}\\
&= -\frac{1}{2\pi i}\oint_{C'} dz\frac{1}{(1+e^{\beta z})}\frac{e^{z\tau}}{(\omega-z)}\\
&= -\operatorname*{Res}_{z=\omega}\frac{e^{z\tau}}{(1+e^{\beta z})(\omega-z)}\\
&= -\frac{e^{\omega\tau}}{1+e^{\beta\omega}}.
\end{aligned}$$

From both results, we can see that the initial sum is odd with respect to $\tau\to\beta-\tau$. Notice the Fermi-Dirac distribution appearance in both results.

If we take the limit $\omega\to 0$, we obtain,

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{-i\omega_n\tau}}{-i\omega_n} = \begin{cases} \frac{1}{\beta}\sum_n\frac{e^{-i\omega_n\tau}}{-i\omega_n} = \frac{1}{2} & ,\ \tau>0\\ \frac{1}{\beta}\sum_n\frac{e^{-i\omega_n(-\tau)}}{-i\omega_n} = -\frac{1}{2} & ,\ \tau<0\end{cases} \tag{46}$$

Thus,

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{-i\omega_n\tau}}{-i\omega_n} = \frac{1}{2}\ sgn(\tau) = G_0^{\beta}(\tau), \tag{47}$$

this means that the Fourier transform (in Matsubara modes) of the free 2-point function is

$$\tilde{G_0}^{\beta}(i\omega_n) = \frac{1}{-i\omega_n}.$$

Another useful example is the sum:

$$
\begin{aligned}
\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{-i\omega_n\tau}}{\omega^2+\omega_n^2} &= \frac{1}{2\omega\beta}\sum_{n=-\infty}^{\infty}\left(\frac{e^{-i\omega_n\tau}}{\omega-i\omega_n}+\frac{e^{-i\omega_n\tau}}{\omega+i\omega_n}\right)\\
&= \frac{1}{2\omega\beta}\sum_{n=-\infty}^{\infty}\left(\frac{e^{-i\omega_n\tau}}{\omega-i\omega_n}+\frac{e^{i\omega_n\tau}}{\omega-i\omega_n}\right)\\
&= \frac{1}{2\omega}\left(\frac{e^{-\omega\tau}}{1+e^{-\beta\omega}}-\frac{e^{\omega\tau}}{1+e^{\beta\omega}}\right)\\
&= \frac{1}{2\omega}\left(\frac{-2\sinh\left(\omega\tau\right)-2\sinh\left[(\tau-\beta)\omega\right]}{4\cosh^2\frac{\beta\omega}{2}}\right)\\
&= \frac{1}{2\omega}\left(\cosh\left(\omega\tau\right)\tanh\left(\frac{\beta\omega}{2}\right)-\sinh\left(\omega\tau\right)\right)
\end{aligned}
$$

where in the second line, we used the fact that the sum is symmetric under $\omega_n \to -\omega_n$ (not symmetric with respect to zero, actually $\omega_0 = -\omega_{-1}$), and in the third line we used the previous examples.

Note. The analogous calculation for the bosonic case is similar, but with frequencies: $\omega_n = \frac{2\pi n}{\beta}$ and using the function $\coth\left(\frac{\beta z}{2}\right)$ instead of $\tanh$.

B.2 Finite Temperature two-point function properties

We already saw that in the free ($J = 0$) case, $G_0^{\beta}(\tau) = \frac{1}{2}\ sgn(\tau)$. Therefore, let us check some important features of the full 2-point function at finite temperature for fermions:

$$
G^{\beta}_{ab}(\tau_1,\tau_2) = \frac{1}{Z}\begin{cases} Tr\left(\chi_a(\tau_1)\chi_b(\tau_2)e^{-\beta H}\right) & , \quad \tau_1 > \tau_2 \\ -Tr\left(\chi_b(\tau_2)\chi_a(\tau_1)e^{-\beta H}\right) & , \quad \tau_1 < \tau_2 \end{cases}, \tag{48}
$$

where $\chi_a(\tau) = e^{\tau H}\chi_a(0)e^{-\tau H}$, being H the Hamiltonian of the theory. This correlator can be shown to be dependent only on one variable $\tau_1 - \tau_2$ (considering $\tau_1 > \tau_2$):

$$
\begin{aligned}
G_{ab}(\tau_1,\tau_2) &= \frac{1}{Z}Tr\left(\chi_a(\tau_1)\chi_b(\tau_2)e^{-\beta H}\right)\\
&= \frac{1}{Z}Tr\left(e^{\tau_1 H}\chi_a(0)e^{-\tau_1 H}e^{\tau_2 H}\chi_b(0)e^{-\tau_2 H}e^{-\beta H}\right)\\
&= \frac{1}{Z}Tr\left(e^{(\tau_1-\tau_2)H}\chi_a(0)e^{-(\tau_1-\tau_2)H}\chi_b(0)e^{-\beta H}\right)\\
&= \frac{1}{Z}Tr\left(\chi_a(\tau_1-\tau_2)\chi_b(0)e^{-\beta H}\right)\\
&= G_{ab}(\tau_1-\tau_2,0)\\
&= G_{ab}(\tau_1-\tau_2)
\end{aligned} \tag{49}
$$

From this, we can derive the Kubo-Martin-Schwinger (KMS) condition for finite-T correlators:

$$
\begin{aligned}
G_{ab}(\tau) &= -\frac{1}{Z} Tr\left(\chi_b(0)\chi_a(\tau)e^{-\beta H}\right), \qquad \tau < 0 \\
&= -\frac{1}{Z} Tr\left(\chi_b(0)e^{\tau H}\chi_a(0)e^{-\tau H}e^{-\beta H}\right) \\
&= -\frac{1}{Z} Tr\left(\chi_b(0)e^{-\beta H}e^{\beta H}e^{\tau H}\chi_a(0)e^{-\tau H}e^{-\beta H}\right) \\
&= -\frac{1}{Z} Tr\left(\chi_b(0)e^{-\beta H}e^{(\beta+\tau)H}\chi_a(0)e^{-(\beta+\tau)H}\right) \\
&= -\frac{1}{Z} Tr\left(\chi_b(0)e^{-\beta H}\chi_a(\beta+\tau)\right) \\
&= -\frac{1}{Z} Tr\left(\chi_a(\beta+\tau)\chi_b(0)e^{-\beta H}\right), \qquad (\tau+\beta > 0) \\
&= -G_{ab}(\beta+\tau)
\end{aligned}
\tag{50}
$$

Applying the latter property to the SYK correlator ($a = b$):

$$
G_\beta(\tau) = -G_\beta(\beta+\tau). \tag{51}
$$

and

$$
\begin{aligned}
G_\beta(\tau) &= \frac{1}{Z} Tr\left(\chi(\tau)\chi(0)e^{-\beta H}\right), \qquad \tau > 0 \\
&= \frac{1}{Z} Tr\left(e^{\tau H}\chi(0)e^{-\tau H}\chi(0)e^{-\beta H}\right) \\
&= \frac{1}{Z} Tr\left(\chi(0)e^{-(\beta-\tau)H}\chi(0)e^{-\tau H}e^{-\beta H}e^{\beta H}\right) \\
&= \frac{1}{Z} Tr\left(e^{(\beta-\tau)H}\chi(0)e^{-(\beta-\tau)H}\chi(0)e^{-\beta H}\right) \\
&= \frac{1}{Z} Tr\left(\chi(\beta-\tau)\chi(0)e^{-\beta H}\right), \qquad \beta-\tau > 0 \\
&= G_\beta(\beta-\tau).
\end{aligned}
\tag{52}
$$

This last result is a consequence of the particle-hole (PH) symmetry in SYK[29]. Therefore, we can summarize both features as:

$$
\begin{aligned}
&G_\beta(\tau) = -G_\beta(\beta+\tau), \qquad (KMS) \\
&G_\beta(\tau) = G_\beta(\beta-\tau), \qquad (PH).
\end{aligned}
\tag{53}
$$

C Spin Glass (SG)

The Spin Glass is a magnetic state characterized by randomness in the alignment of spins and in the couplings. In general, its time dependence distinguishes from other magnetic materials. Besides, its magnetic behavior (magnetization) is as follows, as we send $\vec{B} \to \vec{0}$:

1. Paramagnetic materials: $\vec{M} \to \vec{0}$ in an exponential way.

[29]This is analogous to the Charge conjugation symmetry (C) in high energy physics, and has to do with the fact that the fermions are Majorana, namely, they are their own antiparticle (in other words, the Lagrangian is invariant under the exchange of particles and holes).

2. Ferromagnetic materials: Remains magnetized at a remanent value.

3. Spin glasses: $\vec{M} \to \vec{0}$ (or to a small value) in a non-exponential way.

Spin glass vs. ferromagnetic materials is analogous to a window glass vs. Crystal lattice-based solid:

Ferromagnetic -Spins aligned (same direction).	**Crystal lattice-based solid** -Uniform pattern of atomic bonds.
Spin Glass -Spins aligned randomly.	**Window Glass** -Highly irregular atomic bond structure.

C.1 Sherington-Kirkpatrick model (SK)

The spin glass model:

$$H_{SK} = -\sum_{i<j} J_{ij} S_i S_j, \tag{54}$$

where S_i are the Pauli matrices at the lattice point i, J_{ij} is the coupling of the nearest neighbors i and j (it describes the magnetic nature of the spin-spin interaction, for example a negative J_{ij} value means an anti-ferromagnetic interaction), and the sum is done over all nearest neighbors.

Solving for the free energy, by the replica method, there exists a new magnetic phase below a critical temperature T_c, this is called *spin-glass phase* or *glassy phase*, characterized by a vanishing magnetization[30] and a non-vanishing two-point function at 2 different replicas:

$$\mathcal{Q} = \sum_{i=1}^{N} S_i^\alpha S_i^\beta \neq 0,$$

with $\alpha \neq \beta$ replica indices.

In contrast, SYK does not have a spin-glass phase at low T regime.

D Solution of SD equations in the IR

In this section, we will show that the function

$$G(\tau) = \left(\frac{1}{4\pi J^2}\right)^{1/4} \frac{1}{\sqrt{|\tau|}}\ sgn(\tau) \tag{55}$$

is a solution of SD equations in the IR limit.

$$\begin{aligned} \int d\tau' \Sigma(\tau - \tau') G(\tau') &= \frac{J^2}{4\pi J^2} \int d\tau' \frac{sgn(\tau - \tau')}{\sqrt{|\tau - \tau'|^3}} \frac{sgn(\tau')}{\sqrt{|\tau'|}} \\ &= \frac{1}{4\pi} \int d\omega \tilde{G}(\omega) \tilde{\Sigma}(\omega) e^{-i\omega\tau}, \end{aligned} \tag{56}$$

where

[30] Recall that the magnetization is the first derivative of the Free energy with respect to the field.

$$
\begin{aligned}
\tilde{G}(\omega) &= \int d\tau \ \frac{sgn(\tau)}{\sqrt{|\tau|}} e^{i\omega\tau} \\
&= \int_{-\infty}^{0} d\tau \ \frac{-e^{i\omega\tau}}{\sqrt{-\tau}} + \int_{0}^{\infty} d\tau \ \frac{e^{i\omega\tau}}{\sqrt{\tau}} \\
&= 2\int_{\infty}^{0} dy \ e^{-i\omega y^2} + 2\int_{0}^{\infty} dy \ e^{i\omega y^2} \\
&= 4i\int_{0}^{\infty} dy \ \sin\left(\omega y^2\right) \\
&= \frac{4i}{\sqrt{\omega}}\frac{1}{2}\sqrt{\frac{\pi}{2}} = \frac{2i}{\sqrt{\omega}}\sqrt{\frac{\pi}{2}}.
\end{aligned}
$$

In the last line, we used[31]:

$$
\begin{aligned}
\int_0^\infty \sin\left(x^n\right) dx &= \int_0^\infty du \ \frac{\sin u}{n \ u^{1-1/n}} \\
&= \frac{1}{n} \ Im \ \int_0^\infty du \ u^{1/n-1} \ e^{iu} \\
&= \frac{1}{n} \ Im \ \int_0^\infty dy \ \frac{y^{1/n-1}}{(-i)^{1/n}} \ e^{-y} \\
&= \frac{1}{n} \ Im \left(-\frac{1}{i}\right)^{1/n} \Gamma\left(\frac{1}{n}\right) \\
&= \frac{\Gamma(1/n)}{n} \ Im\left(i^{1/n}\right) \\
&= \frac{\Gamma(1/n)}{n} \ Im\left(e^{i\frac{\pi}{2}\frac{1}{n}}\right) \\
&= \frac{\Gamma(1/n)}{n} \ \sin\left(\frac{\pi}{2n}\right).
\end{aligned} \tag{57}
$$

and

[31] For this calculation, one can also use

$$
\int_{-\infty}^{\infty} e^{iax^2} dx = \sqrt{\frac{\pi}{-ia}}.
$$

However, the result in (57) is more general.

$$
\begin{aligned}
\tilde{\Sigma}(\omega) &= \int d\tau \; \frac{sgn(\tau)}{\sqrt{|\tau|^3}} e^{i\omega\tau} \\
&= \int_{-\infty}^{0} d\tau \; \frac{-e^{i\omega\tau}}{\sqrt{-\tau^3}} + \int_{0}^{\infty} d\tau \; \frac{e^{i\omega\tau}}{\sqrt{\tau^3}} \\
&= 2\int_{\infty}^{0} dy \; \frac{e^{-i\omega y^2}}{y^2} + 2\int_{0}^{\infty} dy \; \frac{e^{i\omega y^2}}{y^2} \\
&= 4i \int_{0}^{\infty} dy \; \frac{\sin\left(\omega y^2\right)}{y^2} \\
&= \omega \frac{4i}{\sqrt{w}} \int_{0}^{\infty} dz \; \frac{\sin\left(z^2\right)}{z^2} \\
&= 4i\sqrt{\omega} \int_{0}^{\infty} dz \int_{0}^{1} da \;\; \cos\left(az^2\right) \\
&= 4i\sqrt{\omega} \int_{0}^{1} da \;\; Re \; \int_{0}^{\infty} dz \; e^{iaz^2} \\
&= 4i\sqrt{\omega} \int_{0}^{1} da \;\; Re \left(\frac{1}{2}\sqrt{\frac{\pi}{-ia}}\right) \\
&= 4i\sqrt{\omega}\frac{1}{2}\sqrt{\pi} \;\; Re(i^{1/2}) \int_{0}^{1} da \; \frac{da}{\sqrt{a}} \\
&= 4i\sqrt{\omega}\frac{1}{2}\sqrt{\pi} \;\; Re(e^{i\frac{\pi}{2}\frac{1}{2}}) 2\sqrt{a}|_{0}^{1} \\
&= 4i\sqrt{\omega}\sqrt{\frac{\pi}{2}}.
\end{aligned}
\tag{58}
$$

Therefore, equation (56) becomes:

$$
\begin{aligned}
\int d\tau' \Sigma(\tau-\tau')G(\tau') &= \frac{J^2}{4\pi J^2} \int d\tau' \; \frac{sgn(\tau-\tau')}{\sqrt{|\tau-\tau'|^3}} \frac{sgn(\tau')}{\sqrt{|\tau'|}} \\
&= \frac{1}{4\pi} \int d\omega \frac{2i}{\sqrt{\omega}} \sqrt{\frac{\pi}{2}} \left(4i\sqrt{\omega}\sqrt{\frac{\pi}{2}}\right) \;\; e^{-i\omega\tau} \\
&= -\frac{1}{4\pi}\frac{8\pi}{2} \int d\omega \; e^{-i\omega\tau} \\
&= -\delta(\tau).
\end{aligned}
\tag{59}
$$

So, expression (55) is a solution of SD equations in the IR.

E Eigenfunctions of the Kernel

The equation that we need to solve is:

$$
\int d\tau_a \; d\tau_b \; v_\alpha(\tau_a, \tau_b) \; K(\tau_a, \tau_b, \tau_3, \tau_4) = g(\alpha) \; v_\alpha(\tau_3, \tau_4),
$$

for the set of eigenvectors

$$
v_\alpha(\tau_a, \tau_b) = \frac{1}{|\tau_a - \tau_b|^{2\alpha}} \; sgn(\tau_a - \tau_b).
$$

This leaves us to solve,

$$\int d\tau_a \int d\tau_b \; \frac{sgn(\tau_a-\tau_b)}{|\tau_a-\tau_b|^{2\alpha}} \; \frac{sgn(\tau_a-\tau_3)}{|\tau_a-\tau_3|^{1/2}} \; \frac{sgn(\tau_b-\tau_4)}{|\tau_b-\tau_4|^{1/2}} \; \frac{sgn(\tau_3-\tau_4)}{|\tau_3-\tau_4|}. \tag{60}$$

Let us use the following relations:

$$\int_{-\infty}^{\xi} d\tau \; \frac{1}{(s-\tau)^x}\frac{1}{(\xi-\tau)^y} = \int_{s}^{\infty} d\tau \; \frac{1}{(\tau-s)^y}\frac{1}{(\tau-\xi)^x} = \frac{1}{(s-\xi)^{x+y-1}} \; \beta(1-y, x+y-1) \tag{61}$$

$$\int_{s}^{\xi} d\tau \; \frac{1}{(\tau-s)^x}\frac{1}{(\xi-\tau)^y} = \frac{1}{(\xi-s)^{x+y-1}} \; \beta(1-x, 1-y), \tag{62}$$

where

$$\beta(x,y) = \int_0^1 dt \; t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \beta(y,x)$$

Hence, dividing the integration region into 8 regions and assuming $\tau_3 > \tau_4$, we have:

i) $\tau_a > \tau_b$, $\tau_a > \tau_3$, $\tau_b > \tau_4$:

$$\begin{aligned}
&\int d\tau_a \int d\tau_b \; \frac{sgn(\tau_a-\tau_b)sgn(\tau_a-\tau_3)sgn(\tau_b-\tau_4)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= \int_{\tau_3}^{\infty} d\tau_a \int_{\tau_4}^{\tau_a} d\tau_b \frac{1}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= \int_{\tau_3}^{\infty} d\tau_a \frac{1}{|\tau_a-\tau_3|^{2\Delta}} \left(\frac{1}{(\tau_a-\tau_4)^{2\Delta+2\alpha-1}} \; \beta(1-2\Delta, 1-2\alpha) \right) \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \; \beta(1-2\Delta, 1-2\alpha) \; \beta(1-2\Delta, 4\Delta+2\alpha-2).
\end{aligned}$$

ii) $\tau_a > \tau_b$, $\tau_a > \tau_3$, $\tau_b < \tau_4$:

$$\begin{aligned}
&\int d\tau_a \int d\tau_b \; \frac{sgn(\tau_a-\tau_b)sgn(\tau_a-\tau_3)sgn(\tau_b-\tau_4)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= \int_{\tau_3}^{\infty} d\tau_a \int_{-\infty}^{\tau_4} d\tau_b \frac{(-1)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= -\int_{\tau_3}^{\infty} d\tau_a \frac{1}{|\tau_a-\tau_3|^{2\Delta}} \frac{1}{(\tau_a-\tau_4)^{2\Delta+2\alpha-1}} \; \beta(1-2\Delta, 2\alpha+4\Delta-1) \\
&= -\frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \; \beta(1-2\Delta, 2\alpha-2\Delta-1) \; \beta(1-2\Delta, 4\Delta+2\alpha-2).
\end{aligned}$$

iii) $\tau_a > \tau_b$, $\tau_a < \tau_3$, $\tau_b > \tau_4$:

$$\begin{aligned}
&\int d\tau_a \int d\tau_b \; \frac{sgn(\tau_a-\tau_b)sgn(\tau_a-\tau_3)sgn(\tau_b-\tau_4)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= \int_{\tau_4}^{\tau_3} d\tau_a \int_{\tau_4}^{\tau_a} d\tau_b \frac{(-1)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= -\int_{\tau_4}^{\tau_3} d\tau_a \frac{1}{|\tau_a-\tau_3|^{2\Delta}} \frac{1}{(\tau_a-\tau_4)^{2\Delta+2\alpha-1}} \; \beta(1-2\Delta, 1-2\alpha) \\
&= -\frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \; \beta(1-2\Delta, 1-2\alpha) \; \beta(1-2\Delta, 2-2\Delta-2\alpha).
\end{aligned}$$

iv) $\tau_a > \tau_b$, $\tau_a < \tau_3$, $\tau_b < \tau_4$:

$$\begin{aligned}
&\int d\tau_a \int d\tau_b \, \frac{sgn(\tau_a-\tau_b)sgn(\tau_a-\tau_3)sgn(\tau_b-\tau_4)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= \int_{-\infty}^{\tau_4} d\tau_a \int_{\tau_b}^{\tau_3} d\tau_b \frac{1}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= \int_{-\infty}^{\tau_4} d\tau_a \frac{1}{|\tau_a-\tau_3|^{2\Delta}} \frac{1}{(\tau_a-\tau_4)^{2\Delta+2\alpha-1}} \; \beta(1-2\alpha, 1-2\Delta) \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \; \beta(1-2\alpha, 1-2\Delta) \; \beta(1-2\Delta, 4\Delta+2\alpha-2).
\end{aligned}$$

v) $\tau_a < \tau_b$, $\tau_a > \tau_3$, $\tau_b > \tau_4$:

$$\begin{aligned}
&\int d\tau_a \int d\tau_b \, \frac{sgn(\tau_a-\tau_b)sgn(\tau_a-\tau_3)sgn(\tau_b-\tau_4)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= \int_{\tau_3}^{\infty} d\tau_b \int_{\tau_3}^{\tau_b} d\tau_a \frac{(-1)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= -\int_{\tau_3}^{\infty} d\tau_b \frac{1}{|\tau_b-\tau_4|^{2\Delta}} \frac{1}{(\tau_b-\tau_3)^{2\Delta+2\alpha-1}} \; \beta(1-2\Delta, 1-2\alpha) \\
&= -\frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \; \beta(1-2\Delta, 1-2\alpha) \; \beta(2-2\alpha-2\Delta, 4\Delta+2\alpha-2).
\end{aligned}$$

vi) $\tau_a < \tau_b$, $\tau_a > \tau_3$, $\tau_b < \tau_4$. This set of inequalities imply $\tau_3 < \tau_a < \tau_b < \tau_4$ contradicting our assumption of $\tau_3 > \tau_4$. So, this region does not exist.

vii) $\tau_a < \tau_b$, $\tau_a < \tau_3$, $\tau_b > \tau_4$:

$$\begin{aligned}
&\int d\tau_a \int d\tau_b \, \frac{sgn(\tau_a-\tau_b)sgn(\tau_a-\tau_3)sgn(\tau_b-\tau_4)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= \begin{cases} \int_{\tau_4}^{\infty} d\tau_b \int_{-\infty}^{\tau_4} d\tau_a \frac{1}{|\tau_a-\tau_b|^{2\alpha}|\tau_a-\tau_3|^{2\Delta}|\tau_b-\tau_4|^{2\Delta}} \\ \int_{\tau_4}^{\tau_3} d\tau_a \int_{\tau_a}^{\infty} d\tau_b \frac{1}{|\tau_a-\tau_b|^{2\alpha}|\tau_a-\tau_3|^{2\Delta}|\tau_b-\tau_4|^{2\Delta}} \end{cases} \\
&= \begin{cases} \int_{-\infty}^{\tau_4} d\tau_a \frac{1}{|\tau_a-\tau_3|^{2\Delta}} \frac{1}{(\tau_4-\tau_a)^{2\Delta+2\alpha-1}} \; \beta(1-2\Delta, 2\alpha+2\Delta-1) \\ \int_{\tau_4}^{\tau_3} d\tau_a \frac{1}{|\tau_a-\tau_3|^{2\Delta}} \frac{1}{(\tau_4-\tau_a)^{2\Delta+2\alpha-1}} \; \beta(1-2\alpha, 2\alpha+2\Delta-1) \end{cases} \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \begin{cases} \beta(1-2\Delta, 2\alpha+2\Delta-1) \; \beta(2-2\alpha-2\Delta, 2\alpha+4\Delta-2) \\ \beta(1-2\alpha, 2\alpha+2\Delta-1) \; \beta(1-2\Delta, 2-2\alpha-2\Delta) \end{cases} .
\end{aligned}$$

viii) $\tau_a < \tau_b$, $\tau_a < \tau_3$, $\tau_b < \tau_4$:

$$\begin{aligned}
&\int d\tau_a \int d\tau_b \, \frac{sgn(\tau_a-\tau_b)sgn(\tau_a-\tau_3)sgn(\tau_b-\tau_4)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= \int_{-\infty}^{\tau_4} d\tau_a \int_{\tau_a}^{\tau_4} d\tau_b \frac{(-1)}{|\tau_a-\tau_b|^{2\alpha}\,|\tau_a-\tau_3|^{2\Delta}\,|\tau_b-\tau_4|^{2\Delta}} \\
&= -\int_{-\infty}^{\tau_4} d\tau_a \frac{1}{|\tau_3-\tau_a|^{2\Delta}} \frac{1}{(\tau_4-\tau_a)^{2\Delta+2\alpha-1}} \; \beta(1-2\alpha, 1-2\Delta) \\
&= -\frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \; \beta(1-2\alpha, 1-2\Delta) \; \beta(2-2\alpha-2\Delta, 4\Delta+2\alpha-2).
\end{aligned}$$

Therefore, (60) reads:

$$\begin{aligned}
&\int d\tau_a \int d\tau_b \; \frac{sgn(\tau_a-\tau_b)}{|\tau_a-\tau_b|^{2\alpha}} \; \frac{sgn(\tau_a-\tau_3)}{|\tau_a-\tau_3|^{1/2}} \; \frac{sgn(\tau_b-\tau_4)}{|\tau_b-\tau_4|^{1/2}} \; \frac{sgn(\tau_3-\tau_4)}{|\tau_3-\tau_4|} \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \left[\beta(1-2\Delta,1-2\alpha)\;\beta(1-2\Delta,4\Delta+2\alpha-2)\right. \\
&\quad -\beta(1-2\Delta,2\alpha-2\Delta-1)\;\beta(1-2\Delta,4\Delta+2\alpha-2) \\
&\quad -\beta(1-2\Delta,1-2\alpha)\;\beta(1-2\Delta,2-2\Delta-2\alpha) \\
&\quad +\beta(1-2\alpha,1-2\Delta)\;\beta(1-2\Delta,4\Delta+2\alpha-2) \\
&\quad -\beta(1-2\Delta,1-2\alpha)\;\beta(2-2\alpha-2\Delta,4\Delta+2\alpha-2) \\
&\quad +\beta(1-2\Delta,2\alpha+2\Delta-1)\;\beta(2-2\alpha-2\Delta,4\Delta+2\alpha-2) \\
&\quad +\beta(1-2\alpha,2\alpha+2\Delta-1)\;\beta(1-2\Delta,2-2\Delta-2\alpha) \\
&\quad \left.-\beta(1-2\alpha,1-2\Delta)\;\beta(2-2\alpha-2\Delta,4\Delta+2\alpha-2)\right] \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \left(2\beta\left(\frac{1}{2},1-2\alpha\right)\left[\beta(1/2,2\alpha-1)-\beta(3/2-2\alpha,2\alpha-1)\right]\right. \\
&\quad +\beta\left(\frac{1}{2},2\alpha-\frac{1}{2}\right)\left[\beta(3/2-2\alpha,2\alpha-1)-\beta(1/2,2\alpha-1)\right] \\
&\quad \left.+\;\beta\left(\frac{1}{2},\frac{3}{2}-2\alpha\right)\left[\beta(1-2\alpha,2\alpha-1/2)-\beta(1/2,1-2\alpha)\right]\right) \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \left[2\frac{\sqrt{\pi}\Gamma(1-2\alpha)}{\Gamma(3/2-2\alpha)}\left(\frac{\sqrt{\pi}\Gamma(2\alpha-1)}{\Gamma(2\alpha-1/2)}-\frac{\Gamma(3/2-2\alpha)\Gamma(2\alpha-1)}{\sqrt{\pi}}\right)\right. \\
&\quad +\frac{\sqrt{\pi}\Gamma(2\alpha-1/2)}{\Gamma(2\alpha)}\left(\frac{\Gamma(3/2-2\alpha)\Gamma(2\alpha-1)}{\sqrt{\pi}}-\frac{\sqrt{\pi}\Gamma(2\alpha-1)}{\Gamma(3/2-2\alpha)}\right) \\
&\quad \left.+\;\frac{\sqrt{\pi}\Gamma(3/2-2\alpha)}{\Gamma(2-2\alpha)}\left(\frac{\Gamma(1-2\alpha)\Gamma(2\alpha-1/2)}{\sqrt{\pi}}-\frac{\sqrt{\pi}\Gamma(1-2\alpha)}{\Gamma(3/2-2\alpha)}\right)\right] \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \left[\frac{2\pi\Gamma(2\alpha-1)\Gamma(1-2\alpha)}{\Gamma(3/2-2\alpha)\Gamma(2\alpha-1/2)}-2\Gamma(1-2\alpha)\Gamma(2\alpha-1)\right] \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \left[\frac{2\pi\Gamma(2\alpha-1)\frac{\pi}{\sin(2\pi\alpha)\Gamma(2\alpha)}}{\Gamma(1-(2\alpha-1/2))\Gamma(2\alpha-1/2)}-2\Gamma(2\alpha-1)\frac{\pi}{\sin(2\pi\alpha)\Gamma(2\alpha)}\right] \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}} \left[\frac{2\pi^2}{\sin 2\pi\alpha(2\alpha-1)}\frac{\sin 2\pi\alpha-\pi/2}{\pi}-\frac{2\pi}{\sin(2\pi\alpha)(2\alpha-1)}\right] \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}}\frac{2\pi}{\sin 2\pi\alpha(2\alpha-1)}\left[-1-\cos(2\pi\alpha)\right] \\
&= \frac{1}{(\tau_3-\tau_4)^{4\Delta+2\alpha-2}}\frac{2\pi}{(1-2\alpha)}\frac{1}{\tan\alpha\pi}.
\end{aligned}$$

Thus, since $\tau_3 > \tau_4$,

$$\begin{aligned}
\int d\tau_a d\tau_b \; v_\alpha(\tau_a,\tau_b)K(\tau_a,\tau_b,\tau_3,\tau_4) &= \frac{(-3J^2)}{4\pi J^2}\frac{1}{(\tau_3-\tau_4)^{1+2\alpha-2}}\frac{2\pi}{(1-2\alpha)}\frac{1}{\tan\alpha\pi}\frac{1}{\tau_{34}} \\
&= -\frac{3}{2}\frac{1}{(1-2\alpha)}\frac{1}{\tan\alpha\pi}\frac{1}{\tau_{34}^{2\alpha}} \\
&= g(\alpha)\; v_\alpha(\tau_3,\tau_4),
\end{aligned} \tag{63}$$

where

$$g(\alpha) = -\frac{3}{2}\frac{1}{(1-2\alpha)}\frac{1}{\tan \alpha\pi}$$

is the eigenvalue of the kernel with eigenfunction $v_\alpha(\tau_a, \tau_b)$.

F Comments on Conformal Field Theory (CFT)

A CFT is a theory invariant under Conformal Group transformations, which are generated by translations, rotations, dilatations and special conformal transformations (SCT), more precisely:

$$\begin{aligned}
x^\mu &\to x'^\mu = x^\mu + a^\mu \qquad (Translations)\\
x^\mu &\to x'^\mu = \omega^\mu{}_\nu x^\nu \qquad (Rotations)\\
x^\mu &\to x'^\mu = \lambda x^\mu \qquad (Scaling)\\
x^\mu &\to x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2} \qquad (SCT),
\end{aligned}$$

where $a^2 = a^\mu a_\mu$, the modulus is

$$x'^2 = \frac{x^2}{1-2b\cdot x + b^2x^2},$$

and considering two different space-time points x_1 and x_2:

$$\begin{aligned}
(x'_1 - x'_2)^2 &= x'^2_1 - 2x'_1\cdot x'_2 + x'^2_2\\
&= \frac{x_1^2(1-2b\cdot x_2 + b^2x_2^2) - 2\left[x_1\cdot x_2 - (b\cdot x_1)x_2^2 - (b\cdot x_2)x_1^2 + b^2x_1^2x_2^2\right]}{(1-2b\cdot x_1 + b^2x_1^2)(1-2b\cdot x_2+b^2x_2^2)}\\
&\qquad + \frac{x_2^2(1-2b\cdot x_1 + b^2x_1^2)}{(1-2b\cdot x_1 + b^2x_1^2)(1-2b\cdot x_2+b^2x_2^2)}\\
&= \frac{x_1^2 - 2x_1\cdot x_2 + x_2^2}{(1-2b\cdot x_1 + b^2x_1^2)(1-2b\cdot x_2+b^2x_2^2)}\\
&= \frac{(x_1-x_2)^2}{(1-2b\cdot x_1 + b^2x_1^2)(1-2b\cdot x_2+b^2x_2^2)},
\end{aligned}$$

therefore

$$|x'_1 - x'_2| = \frac{|x_1-x_2|}{(1-2b\cdot x_1 + b^2x_1^2)^{1/2}(1-2b\cdot x_2+b^2x_2^2)^{1/2}}. \tag{64}$$

F.1 CFT for $d \geq 3$

Starting with $d \geq 3$ (the case $d = 2$ is particularly interesting and will be studied separately[32] Let us start by counting the number of generators, since $\mathrm{CFT}_{d-1,1}$ has[33]:

[32]The case $d = 1$ also falls into this section and corresponds to the case of Conformal Quantum Mechanics since time is the only dimension.

[33]Recall that the $SO(n)$ group has dimension (= number of (anti-symmetric) generators) $\frac{n(n-1)}{2}$.

$$1\,(dilation) + d\,(translations) + d\,(SCT) + \frac{d(d-1)}{2}\,(rotations) = \frac{(d+2)(d+1)}{2}\,CFT\ generators.$$

Therefore, this tells us that the conformal group in $(d-1,1)$ dimensions is isomorphic to the group $SO(d+1,1)$. Before studying the properties of the correlation functions, we have to define the scaling dimension Δ of a field operator Φ as the action of the scaling transformation on it, that is

$$\Phi(\lambda x) = \lambda^{-\Delta}\Phi(x).$$

Now, if the metric transforms in a covariant way, namely

$$g_{\mu\nu} \to g'_{\mu\nu} = \Lambda(x) g_{\mu\nu},$$

to fix the value of $\Lambda(x)$, we know that

$$x^{\mu} \to x'^{\mu} = \Lambda(x) x^{\mu}$$

so that $\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Lambda(x)\delta^{\mu}{}_{\nu}$. So that the determinant of the Jacobian of the conformal group transformations gives:

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda(x)^{d}.$$

Thus, the transformation of a scalar field ϕ under the conformal group is:

$$\phi(x) \to \phi'(x') = \Lambda(x)^{-\Delta}\phi(x) = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d}\phi(x). \tag{65}$$

Fields that transform in such a way (covariantly) are called *quasi-primary* fields.

F.1.1 Conformal Invariants

Consider any set of points in a $CFT_{d+1,1}$, then

i) Translation and rotation invariance implies

$$|x_1 - x_2| = invariant.$$

ii) Including scale invariance requires

$$\frac{|x_1 - x_2|}{|x_3 - x_4|} = invariant.$$

iii) If we apply SCT to $|x_1 - x_2|$, then by (64)

$$|x_1 - x_2| \to \frac{|x_1 - x_2|}{(1 - 2b\cdot x_1 + b^2 x_1^2)^{1/2}(1 - 2b\cdot x_2 + b^2 x_2^2)^{1/2}}.$$

So, the conformal invariant is composed by cross ratios, for instance:

$$\frac{|x_1 - x_2|\,|x_3 - x_4|}{|x_1 - x_3|\,|x_2 - x_4|},$$

where it is important to note that there are $\frac{n}{2}(n-3)$ cross ratios for n distinct points[34] (the expressions can be quite complicated).

[34] For the two independent ratios

$$\frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}, \qquad \frac{x_{ij}x_{kl}}{x_{il}x_{jk}}, \qquad i,j,k,l\ all\ distinct$$

F.1.2 Correlation functions and its constraints

In general, the n-point correlation function is given by calculating the path integral

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \frac{1}{Z}\int [\mathcal{D}\Phi]\Phi_1(x_1)\cdots\Phi_n(x_n)e^{-S[\Phi]}, \tag{66}$$

where $Z = \int[\mathcal{D}\Phi]e^{-S[\Phi]}$.

By conformal transformations we can see that

$$\begin{aligned}
\langle \phi_1(x_1')\cdots\phi_n(x_n')\rangle &= \frac{1}{Z}\int [\mathcal{D}\Phi']\Phi_1'(x_1')\cdots\Phi_n'(x_n')e^{-S[\Phi']} \\
&= \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{-\Delta_1/d}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{-\Delta_n/d}\frac{1}{Z}\int[\mathcal{D}\Phi]\Phi_1(x_1)\cdots\Phi_n(x_n)e^{-S[\Phi]} \\
&= \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{-\Delta_1/d}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{-\Delta_n/d}\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle,
\end{aligned}$$

where in the second line both the action and the functional measure are conformally invariant. Equivalently,

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d}\langle \phi_1(x_1')\cdots\phi_n(x_n')\rangle. \tag{67}$$

Start with 1-point function. If we apply the invariance under rotations and translations, we then have

$$\langle\phi(x)\rangle = h\left(|x|\right),$$

now using the rescaling covariance, we require

$$h\left(|x|\right) = \lambda^{\Delta}h\left(\lambda|x|\right)$$

to fulfill this condition, $h(|x|)$ should be a polynomial in $|x|$, namely

$$\langle\phi(x)\rangle = \frac{C}{|x|^{\Delta}}.$$

Finally, let's use (64) and (67):

$$\frac{C}{|x|^{\Delta}} = \frac{C}{|x|^{\Delta}}\frac{(1-2b\cdot x+b^2x^2)^{\Delta/2}}{(1-2b\cdot x+b^2x^2)^{\Delta}},$$

since Δ can have any value (scaling dimension of the field), this expression is satisfied only if $C = 0$. Hence,

$$\langle\phi(x)\rangle = 0. \tag{68}$$

Let us see what happens to the two point function. If we apply the invariance under rotations and translations, we then have

we have $\binom{n}{2} = n(n-1)/2$ possible x_{ij} terms and $\binom{n-2}{2} = (n-2)(n-3)/2$ possible x_{kl}, then since the denominator is fixed once chosen the numerator, we have to divide by all possibilities of the denominator (to not overcount) which are $n-1$ for fixed i and $n-2$ for fixed j. So the final expression is $n(n-3)/2$.

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = h\left(|x_1 - x_2|\right),$$

now using the rescaling covariance, we require

$$h\left(|x_1 - x_2|\right) = \lambda^{\Delta_1+\Delta_2} h\left(\lambda|x_1 - x_2|\right)$$

to fulfill this condition, $h(|x|)$ should be a polynomial in $|x|$, namely

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{\Delta_1\Delta_2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}.$$

Finally, let's use (64) and (67):

$$\frac{C_{\Delta_1\Delta_2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{\Delta_1\Delta_2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \frac{(1-2b\cdot x_1 + b^2x_1^2)^{(\Delta_1+\Delta_2)/2}(1-2b\cdot x_2 + b^2x_2^2)^{(\Delta_1+\Delta_2)/2}}{(1-2b\cdot x_1 + b^2x_1^2)^{\Delta_1}(1-2b\cdot x_2 + b^2x_2^2)^{\Delta_2}},$$

from this last expression, for $C_{\Delta_1\Delta_2} \neq 0$, we have $\Delta_1 + \Delta_2 = 2\Delta_1$ and $\Delta_1 + \Delta_2 = 2\Delta_2$, that is $\Delta_1 = \Delta_2$. Notice that we can rescale the operators ϕ in such a way that the coefficient $C_{\Delta_1\Delta_2} = 1$. In conclusion, the 2-point function in a CFT for a (quasi-primary) scalar field is

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta_1}}\ \delta_{\Delta_1,\Delta_2}. \tag{69}$$

Finally, let's study in a similar fashion the 3-point function. So, covariance under rotations, translations and scaling gives

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \sum_{\substack{a,b,c \\ a+b+c=\Delta_1+\Delta_2+\Delta_3}} \frac{C^{abc}_{\Delta_1\Delta_2\Delta_3}}{|x_1 - x_2|^a\,|x_2 - x_3|^b\,|x_1 - x_3|^c},$$

using (64) and (67), we have

$$\frac{C^{abc}_{\Delta_1\Delta_2\Delta_3}}{|x_1 - x_2|^a\,|x_2 - x_3|^b\,|x_1 - x_3|^c} = \frac{C^{abc}_{\Delta_1\Delta_2\Delta_3}}{|x_1 - x_2|^a\,|x_2 - x_3|^b\,|x_1 - x_3|^c}\frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}},$$

where $\gamma_i = 1 - 2b\cdot x_i + b^2x_i^2$. Then, for non-zero $C^{abc}_{\Delta_1\Delta_2\Delta_3}$, we require

$$a + c = 2\Delta_1, \qquad a + b = 2\Delta_2, \qquad b + c = 2\Delta_3, \qquad a + b + c = \Delta_1 + \Delta_2 + \Delta_3,$$

whose unique solution is $a = \Delta_1 + \Delta_2 - \Delta_3$, $b = \Delta_2 + \Delta_3 - \Delta_1$, $c = \Delta_1 + \Delta_3 - \Delta_2$. Hence,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{\Delta_1\Delta_2\Delta_3}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}\,|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}\,|x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}}. \tag{70}$$

Note that $C_{\Delta_1\Delta_2\Delta_3}$ is unique and cannot be fixed to be 1 since the fields were already rescaled to fix $C_{\Delta_1\Delta_2}$ to unity.

If we want to calculate n-point functions for $n \geq 4$, we have to consider the $n(n-3)/2$ cross-ratios, since they are invariants of the conformal group[35]. So, for the 4-point function, the best we can do is to argue that it has the form:

[35] For the $n < 4$ case, we cannot form cross ratios, so the procedure is the one followed before.

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = h\left(\frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|}, \frac{|x_{12}||x_{34}|}{|x_{14}||x_{23}|}\right)\prod_{i<j}^{4}|x_{ij}|^{\Delta/3-\Delta_i-\Delta_j}, \tag{71}$$

where $\Delta = \sum_i^4 \Delta_i$, $x_{ij} = x_i - x_j$ and the function $h(u,v)$ is not simply fixed by conformal invariance.

F.2 CFT for $d = 2$

MAIN COMMENTS ON THIS!!

G Two-point function corrections

In this section, we calculate in detail the corrections of $G(\tau)$ from the perturbed action $I_{SYK} = I_{CFT} + I_{g,\mathcal{O}}$. So, let us start with the first order correction:

$$\delta^{(1)}G_h(\tau_{12}) = -g_h\int_{-\infty}^{\infty} d\tau_3 \frac{c_h b^\Delta sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta-h}|J\tau_{13}|^h|J\tau_{23}|^h}.$$

Without loss of generality, let us assume that $\tau_1 > \tau_2$, then the possible integrals we can have are:

- If $\tau_2 > \tau_3$, the integral becomes,

$$\begin{aligned}
-g_h\int_{-\infty}^{\infty} d\tau_3 \frac{c_h b^\Delta sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta-h}|J\tau_{13}|^h|J\tau_{23}|^h} &= -\frac{g_h c_h b^\Delta}{J^{2\Delta+h}}\int_{-\infty}^{\tau_2} d\tau_3 \frac{1}{(\tau_{12})^{2\Delta-h}(\tau_{13})^h(\tau_{23})^h}\\
&= -\frac{g_h c_h b^\Delta}{J^{2\Delta+h}(\tau_{12})^{2\Delta-h}}\int_{-\infty}^{\tau_2} d\tau_3 \frac{1}{\tau_{13}^h\tau_{23}^h}\\
&= -\frac{g_h c_h b^\Delta}{J^{2\Delta+h}\tau_{12}^{2\Delta+h-1}}\beta(1-h, 2h-1)\\
&= -\frac{g_h c_h b^\Delta J^{-1}}{(J\tau_{12})^{2\Delta}(J\tau_{12})^{h-1}}\beta(1-h, 2h-1).
\end{aligned}$$

- If $\tau_2 < \tau_3$ and $\tau_1 > \tau_3$, we have,

$$\begin{aligned}
-g_h\int_{-\infty}^{\infty} d\tau_3 \frac{c_h b^\Delta sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta-h}|J\tau_{13}|^h|J\tau_{23}|^h} &= -\frac{g_h c_h b^\Delta}{J^{2\Delta+h}}\int_{\tau_2}^{\tau_1} d\tau_3 \frac{1}{(\tau_{12})^{2\Delta-h}(\tau_{13})^h(\tau_{32})^h}\\
&= -\frac{g_h c_h b^\Delta}{J^{2\Delta+h}(\tau_{12})^{2\Delta-h}}\int_{\tau_2}^{\tau_1} d\tau_3 \frac{1}{\tau_{13}^h\tau_{32}^h}\\
&= -\frac{g_h c_h b^\Delta}{J^{2\Delta+h}\tau_{12}^{2\Delta+h-1}}\beta(1-h, 1-h)\\
&= -\frac{g_h c_h b^\Delta J^{-1}}{(J\tau_{12})^{2\Delta}(J\tau_{12})^{h-1}}\beta(1-h, 1-h).
\end{aligned}$$

- If $\tau_3 > \tau_1 > \tau_2$, the integral becomes,

$$
\begin{aligned}
-g_h \int_{-\infty}^{\infty} d\tau_3 \frac{c_h b^\Delta sgn(\tau_{12})}{|J\tau_{12}|^{2\Delta-h}|J\tau_{13}|^h|J\tau_{23}|^h} &= -\frac{g_h c_h b^\Delta}{J^{2\Delta+h}} \int_{\tau_1}^{\infty} d\tau_3 \frac{1}{(\tau_{12})^{2\Delta-h}(\tau_{31})^h(\tau_{32})^h} \\
&= -\frac{g_h c_h b^\Delta}{J^{2\Delta+h}(\tau_{12})^{2\Delta-h}} \int_{\tau_1}^{\infty} d\tau_3 \frac{1}{\tau_{31}^h \tau_{32}^h} \\
&= -\frac{g_h c_h b^\Delta}{J^{2\Delta+h}\tau_{12}^{2\Delta+h-1}} \beta(1-h, 2h-1) \\
&= -\frac{g_h c_h b^\Delta J^{-1}}{(J\tau_{12})^{2\Delta}(J\tau_{12})^{h-1}} \beta(1-h, 2h-1),
\end{aligned}
$$

where to solve the integrals, we made use of (61) and (62). Thus,

$$
\begin{aligned}
\delta^{(1)} G_h(\tau_{12}) &= -\frac{g_h c_h b^\Delta J^{-1}}{|J\tau_{12}|^{2\Delta}|J\tau_{12}|^{h-1}} \left[2\beta(1-h, 2h-1) + \beta(1-h, 1-h)\right] \\
&= -G_c(\tau_{12}) \frac{g_h c_h J^{-1}}{|J\tau_{12}|^{h-1}} \left[\frac{2\Gamma(1-h)\Gamma(2h-1)}{\Gamma(h)} + \frac{\Gamma(1-h)\Gamma(1-h)}{\Gamma(2-2h)}\right] \\
&= -G_c(\tau_{12}) \frac{g_h c_h J^{-1}}{|J\tau_{12}|^{h-1}} \Gamma(1-h) \left[\frac{2\Gamma(2h)}{(2h-1)\Gamma(h)} - \frac{\pi}{\sin\pi h} \frac{\sin 2\pi h}{\pi(2h-1)} \frac{\Gamma(2h)}{\Gamma(h)}\right] \\
&= -G_c(\tau_{12}) \frac{g_h c_h J^{-1}}{|J\tau_{12}|^{h-1}} \frac{\pi}{\sin\pi h} \frac{\Gamma(2h)}{(2h-1)\Gamma(h)^2} \left[2 - \frac{\sin 2\pi h}{\sin \pi h}\right] \\
&= -G_c(\tau_{12}) \frac{g_h c_h J^{-1}}{|J\tau_{12}|^{h-1}} \frac{2\pi}{\sin\pi h} \frac{\Gamma(2h)}{(2h-1)\Gamma(h)^2} \left[1 - \cos \pi h\right] \\
&= -G_c(\tau_{12}) \frac{g_h c_h J^{-1}}{|J\tau_{12}|^{h-1}} \frac{\pi}{2\sin\frac{\pi h}{2}\cos\frac{\pi h}{2}} \frac{\Gamma(2h)}{(h-1/2)\Gamma(h)^2} \left[2\sin^2\frac{\pi h}{2}\right] \\
&= -G_c(\tau_{12}) \frac{g_h c_h J^{-1}}{|J\tau_{12}|^{h-1}} \frac{\Gamma(2h)}{(h-1/2)\Gamma(h)^2} \pi \tan\frac{\pi h}{2} \\
&= -G_c(\tau_{12}) \frac{\alpha_h}{|J\tau_{12}|^{h-1}},
\end{aligned}
$$

where $\alpha_h = g_h c_h J^{-1} \frac{\Gamma(2h)}{(h-1/2)\Gamma(h)^2} \pi \tan\frac{\pi h}{2}$, or replacing c_h:

$$
\alpha_h^2 = \frac{g_h^2}{J^2} \frac{1}{(q-1)b\ k'(h)} \frac{\pi \tan\frac{\pi h}{2}}{(h-1/2)} \frac{\Gamma(2h)}{\Gamma(h)^2}.
$$

Hence,

$$
G(\tau) = G_c(\tau)\left[1 - \sum_h \frac{\alpha_h}{|J\tau|^{h-1}} + \cdots\right].
$$

For the second order corrections, we need to solve

$$
\delta^{(2)} G_{h_1,h_2}(\tau_{12}) = \sum_h \int_{-\infty}^{\infty} d\tau_3 \int_{-\infty}^{\infty} d\tau_4 \frac{c_h c_{h,h_1,h_2} b^\Delta sgn(\tau_{12})|\tau_{14}|^{h_{12}}}{|J\tau_{12}|^{2\Delta}|J\tau_{34}|^{h_1+h_2}|\tau_{13}|^{h_{12}}}\ z^h\ {}_2F_1(h, h+h_{12}, 2h, z),
$$

where $z = \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}$ and $h_{12} = h_1 - h_2$. The OPE coefficients c_h and c_{h,h_1,h_2} are fixed by the 3-point functions $\langle \chi\chi\mathcal{O}_h\rangle$ and $\langle \mathcal{O}_{h_1}\mathcal{O}_{h_2}\mathcal{O}_h\rangle$, respectively. Therefore, according to Tarnopolsky, et al:

$$
\delta^{(2)} G_{h_1,h_2}(\tau) = -G_c(\tau) \frac{a_{h_1 h_2}\alpha_{h_1}\alpha_{h_2}}{|J\tau|^{h_1+h_2-2}}.
$$

Leading us to:

$$G(\tau) = G_c(\tau)\left[1 - \sum_m \frac{\alpha_m}{|J\tau|^{h_m-1}} - \sum_{m,n} \frac{a_{mn}\alpha_m\alpha_n}{|J\tau|^{h_m+h_n-2}} - \sum_{m,n,p} \frac{a_{mnp}\alpha_m\alpha_n\alpha_p}{|J\tau|^{h_m+h_n+h_p-3}} - \cdots\right], \tag{72}$$

where the first values for a_{mn} and a_{mnp} are

$$a_{00} = \frac{(2\Delta+1)(2-2\Delta-\cos 2\pi\Delta)}{8\Delta\cos^2\pi\Delta} \qquad a_{000} = \frac{(\Delta+1)(2\Delta+1)(6\Delta-8+\cos 2\pi\Delta)}{24\Delta^2\cos^2\pi\Delta}.$$

H OPE expansion

The Operator Product Expansion (OPE) is an axiom used to define the product of fields as a sum over the same fields. It offers a non-perturbative approach to QFT. Explicitly, it reads:

$$\begin{aligned} \mathcal{O}_{h_1}(\tau_1)\mathcal{O}_{h_2}(\tau_2) &= \sum_h \frac{C_{h_1h_2h}}{|\tau_{12}|^{h_1+h_2-h}}\left(1 + \frac{1}{2}\tau_{12}\,\partial_{\tau_2} + \cdots\right)\mathcal{O}_h(\tau_2) \\ &= \sum_h \frac{C_{h_1h_2h}}{|\tau_{12}|^{h_1+h_2-h}}\,\mathcal{C}_{h_1h_2h}(\tau_{12},\partial_{\tau_2})\mathcal{O}_h(\tau_2). \end{aligned} \tag{73}$$

That is, it is a sum of primaries and descendants (h runs over all the dimensions including the identity operator, i.e. $h = 0$). The coefficients $\mathcal{C}_{h_1h_2h}(\tau_{12},\partial_{\tau_2})$ are the generators of the descendants and, together with $C_{h_1h_2h}$, are fixed by the functional form of the three-point function.

As an example, that we need, the two-point function G is then written (assuming $\tau_1 > \tau_2$):

$$\begin{aligned} G_\beta(\tau_{12}) &= \frac{1}{N}\left\langle \mathrm{T}\chi_i(\tau_1)\chi_i(\tau_2)\right\rangle_\beta = \frac{1}{N}\left\langle \chi_i(\tau_1)\chi_i(\tau_2)\right\rangle_\beta \\ &= \sum_h \frac{c_h\ sgn(\tau_{12})}{|\tau_{12}|^{2\Delta-h}}\ \left\langle \mathcal{C}_h(\tau_{12},\partial_{\tau_2})\mathcal{O}_h(\tau_2)\right\rangle_\beta. \end{aligned} \tag{74}$$

If we take $\tau_1 = \tau$ and $\tau_2 = 0$, then

$$\begin{aligned} G_\beta(\tau) &= \sum_h \frac{c_h\ sgn(\tau)}{|\tau|^{2\Delta-h}}\,\mathcal{C}_h(\tau)\left\langle \mathcal{O}_h\right\rangle_\beta \\ &= \frac{b^\Delta\ sgn(\tau)}{|J\tau|^{2\Delta}}\sum_h c_h\ |J\tau|^h\left\langle \mathcal{O}_h\right\rangle_\beta. \end{aligned} \tag{75}$$