Complex SYK Comments

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This series of notes are compiled from 2 main papers: Gu, Kitaev, Sachdev and Tarnopolsky (Notes on the Complex SYK) and Bulycheva (A Note on the Complex SYK model with fermions).

1 The model

The complex Sachdev-Ye-Kitaev (cSYK) model is a generalization of the Majorana SYK one, where the N fermions are now complex operators (fields). For $q-$interactions we have that the Hamiltonian has the form:

$$H_{cSYK} = \sum_{\substack{j_1<\cdots<j_{q/2},\\ k_1<\cdots<k_{q/2}}} J_{j_1\dots j_{q/2},k_1\dots k_{q/2}} \mathcal{A}\left\{\psi^\dagger_{j_1}\dots\psi^\dagger_{j_{q/2}}\psi_{k_1}\dots\psi_{k_{q/2}}\right\}, \tag{1}$$

where $\mathcal{A}\{\dots\}$ denotes the antisymmetrized product of operators, the couplings $J_{j_1\dots j_{q/2},k_1\dots k_{q/2}}$ have zero mean and variance:

$$\overline{\left|J_{j_1\dots j_{q/2}k_1\dots k_{q/2}}\right|^2} = J^2\frac{(q/2)!(q/2-1)!}{N^{q-1}}$$

and the operators ϕ_j obey the anti-commutation relations:

$$\left\{\psi^\dagger_j,\psi_k\right\} = \psi^\dagger_j\psi_k + \psi_k\psi^\dagger_j = \delta_{jk}. \tag{2}$$

The antisymmetrized Hamiltonian makes the particle-hole symmetry explicit, i.e. H_{cSYK} is invariant under $\psi_j \leftrightarrow \psi^\dagger_j$. Furthermore, there is an additional symmetry that the Majorana SYK does not possess: $U(1)$ global. To see this, we perform the global transformation of the fields $\psi_j \to e^{i\alpha}\psi_j$ and $\psi^\dagger_j \to e^{-i\alpha}\psi^\dagger_j$. Since this model is one-dimensional (no space), then the corresponding conserved charge is

$$\begin{aligned}
Q &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}_j}\frac{\partial\psi_j}{\partial\alpha} + \frac{\partial\mathcal{L}}{\partial\dot{\psi}^\dagger_j}\frac{\partial\psi^\dagger_j}{\partial\alpha}\\
&= i\psi^\dagger_j\psi_j - i\psi_j\psi^\dagger_j = i\mathcal{A}\left\{\psi^\dagger_j\psi_j\right\}\\
&= 2i\left(\psi^\dagger_j\psi_j - \frac{N}{2}\right)\\
\implies \hat{Q} &= \psi^\dagger_j\psi_j - \frac{N}{2}.
\end{aligned}$$

This charge is related to the UV asymmetry of the two-point function, to show this consider $\tau_1 - \tau_2 = \tau$. In the UV regime ($|\tau| \ll 1/J$), the theory is free and the Hamiltonian is zero (non-interacting case), so $\psi_j(\tau) = e^{\tau H}\psi_j(0)e^{-\tau H} = \psi_j(0) = \psi_j$. Therefore, the zero-temperature free two-point function reads

$$
\begin{aligned}
G_0(\tau_1,\tau_2) &= G_0(\tau_1-\tau_2) = G_0(\tau) \\
&= -\frac{1}{N}\sum_{j,k}\langle \mathrm{T}\psi_j(\tau)\psi_k^\dagger(0)\rangle \\
&= -\frac{1}{N}\sum_{j,k}\left[\Theta(\tau)\,\langle\psi_j\psi_k^\dagger\rangle - \Theta(-\tau)\,\langle\psi_k^\dagger\psi_j\rangle\right] \\
&= \frac{1}{N}\sum_{j,k}\left[\langle\psi_k^\dagger\psi_j\rangle\,(\Theta(-\tau)+\Theta(\tau)) - \Theta(\tau)\delta_{jk}\right] \\
&= \frac{\langle\hat{Q}\rangle}{N} + \frac{1}{2} - \Theta(\tau) \\
\implies G_0(\tau) &= \frac{\langle\hat{Q}\rangle}{N} - \frac{sgn(\tau)}{2}.
\end{aligned}
$$

Therefore, the exact two-point function has as UV limits[1]: $G(0^+) = \mathcal{Q} - \frac{1}{2}$ and $G(0^-) = \mathcal{Q} + \frac{1}{2}$, with $\mathcal{Q} = \frac{\langle\hat{Q}\rangle}{N}$. Which is the same as writing:

$$
G(0^+) = -\frac{1}{2} + \mathcal{Q} \qquad G(\beta^-) = -\frac{1}{2} - \mathcal{Q}, \tag{3}
$$

where we can see the UV asymmetry in an explicit form.

Let us focus on the $q = 4$ case, the model reduces to:

$$
H_{cSYK} = \sum_{\substack{j_1<j_2,\\ k_1<k_2}} J_{j_1j_2,k_1k_2}\;\mathcal{A}\left\{\psi_{j_1}^\dagger\psi_{j_2}^\dagger\psi_{k_1}\psi_{k_2}\right\}, \tag{4}
$$

2 Thermodynamics of cSYK

First, the effective action for this model is now[2]:

$$
\begin{aligned}
I_{cSYK} \equiv \frac{I_{eff}}{N} = &-\log\det\left(-\partial_{\tau_1}\delta(\tau_1,\tau_2) + \mu\delta(\tau_1,\tau_2) - \Sigma(\tau_1,\tau_2)\right) \\
&- \int d\tau_1 d\tau_2\left[\Sigma(\tau_1,\tau_2)G(\tau_2,\tau_1) + \frac{J^2}{q}\left(-G(\tau_1,\tau_2)G(\tau_2,\tau_1)\right)^{\frac{q}{2}}\right].
\end{aligned} \tag{5}
$$

Note. *The complex-SYK action seems to be twice the Majorana SYK one. Naively, one can see this relation as follows:* $N_{SYK} = 2N_{cSYK}$ *(every field in cSYK has two real components),*

[1]In this case, we calculated for the zero-T case, although for finite temperatures the behavior is the same, see Appendix A

[2]The logarithmic part comes from recalling that for two independent Grassmann variables $\bar{\psi}$ and ψ

$$
\int d\bar{\psi}d\psi e^{-a\bar{\psi}\psi} = \int d\bar{\psi}d\psi(1 - a\bar{\psi}\psi) = a.
$$

Also notice that the $-\partial_\tau$ comes from the convention in the definition of the two-point function, in this case $G(\tau_1,\tau_2) = -\frac{1}{N}\sum_j \psi_j(\tau_1)\psi_j^\dagger(\tau_2)$.

then $N_{SYK}I_{SYK} = N_{cSYK}I_{cSYK}$, which implies $I_{cSYK} = 2I_{SYK}$ and there is no $1/2$ factor in front of the cSYK action as it is in the Majorana SYK one.

From this, we can obtain the Schwinger-Dyson (SD) equations, which again in the large N limit are dominated by the saddle point solution[3]:

$$\begin{aligned} \partial_\tau G(\tau) - \mu G(\tau) + \int d\tau' \Sigma(\tau - \tau')G(\tau') &= -\delta(\tau) \\ \Sigma(\tau) &= J^2 G(\tau)^{\frac{q}{2}} G(\beta - \tau)^{\frac{q}{2}-1}, \end{aligned} \tag{6}$$

where we have used $G(\beta - \tau) = -G(-\tau)$ (KMS condition).

From this action, we can have the partition function $-\log Z = \frac{I_{eff}}{N}$ and all the thermodynamics of the model. Therefore, since I_{eff} is only a function of βJ, then $\beta\partial_\beta = J\partial_J$, so that

$$\begin{aligned} -\beta E &= \beta\partial_\beta(-\beta\Phi/N) = J\partial_J(-\beta\Phi/N) = J\partial_J(-I_{eff}/N) \\ &= 2\frac{J^2}{q}\int d\tau_1 d\tau_2 \left(-G(\tau_1,\tau_2)G(\tau_2,\tau_1)\right)^{\frac{q}{2}} \\ &= 2\frac{J^2\beta}{q}\int_0^\beta d\tau \left(-G(\tau)G(-\tau)\right)^{\frac{q}{2}} \\ &= 2\frac{J^2\beta}{q}\int_0^\beta d\tau \left(G(\tau)G(\beta-\tau)\right)^{\frac{q}{2}} \\ &= -\frac{2\beta}{q}\lim_{\tau\to 0^+}\left[\left(\mu - \partial_\tau\right)G(\tau)\right] \\ \implies E &= \frac{2}{q}\lim_{\tau\to 0^+}\left[\left(\mu - \partial_\tau\right)G(\tau)\right], \end{aligned} \tag{7}$$

where $\frac{\Phi}{N} = -\frac{1}{\beta}\log Z = \frac{I_{eff}}{\beta N}$ is the Grand potential per species and in the last line we used SD equations (for $\tau \neq 0$) in the form:

$$J^2 \int d\tau' G(\tau - \tau')^{\frac{q}{2}}(-G(\tau' - \tau))^{\frac{q}{2}-1}G(\tau') = -\partial_\tau G(\tau) + \mu G(\tau),$$

so that if $\tau \to 0^+$:

$$-J^2 \int d\tau'(-G(-\tau'))^{\frac{q}{2}} G(\tau')^{\frac{q}{2}} = \lim_{\tau\to 0^+}\left[\mu G(\tau) - \partial_\tau G(\tau)\right].$$

Using (3), we can write the energy per unit coupling as

$$\epsilon \equiv \frac{E}{J} = \frac{2}{Jq}\left[\mu G(0^+) - \lim_{\tau\to 0^+}\partial_\tau G(\tau)\right] = \frac{2}{Jq}\left[\mu\left(\mathcal{Q} - \frac{1}{2}\right) - \lim_{\tau\to 0^+}\partial_\tau G(\tau)\right]. \tag{8}$$

As in the Majorana SYK case, $\log Z = \phi \equiv -\frac{\beta\Phi}{N} = -\frac{I_{eff}}{N}$ is only a function of βJ and thus

$$\beta E = -\beta\partial_\beta\phi(\beta J) = -J\partial_J\phi(\beta J) = -J\beta\phi'(\beta J),$$

or equivalently,

$$\epsilon = -\phi'(\beta J).$$

[3] A small note in the derivation is that the saddle point solutions are functional derivatives, that is

$$\frac{\delta G(\tau_1,\tau_2)}{\delta G(\tau_3,\tau_4)} = \delta(\tau_1 - \tau_3)\delta(\tau_2 - \tau_4).$$

3 Conformal two-point function

Focusing in the IR regime of the SD equations (6), we can safely ignore the UV term $\sigma(\tau_1, \tau_2) = \delta'(\tau_1, \tau_2) - \mu\delta(\tau_1, \tau_2)$ and obtain the conformal solution in the zero-temperature case:

$$G_c^0(\pm\tau) = \mp e^{\pm\pi\mathcal{E}} b^{\Delta} (J\tau)^{-2\Delta}, \tag{9}$$

where the parameter $\mathcal{E}$ depends implicitly on μ and takes into account the asymmetry of the two-point function solution and $J|\tau| \gg 1$. In a similar fashion, we can introduce the parameter θ that plays the same role as $\mathcal{E}$ but in the frequency domain ($0 < \omega \ll J$, IR limit):

$$\tilde{G}_c^0(\pm i\omega) = \mp i e^{\mp i\theta} \sqrt{\frac{\Gamma(2-2\Delta)}{\Gamma(2\Delta)}}\, b^{\Delta-\frac{1}{2}} \left(\frac{\omega}{J}\right)^{2\Delta-1}.$$

These two spectral asymmetry parameters are related by the following relation:

$$e^{-2i\theta} = \frac{\cos\pi(\Delta + i\mathcal{E})}{\cos\pi(\Delta - i\mathcal{E})}, \qquad e^{2\pi\mathcal{E}} = \frac{\sin(\pi\Delta + \theta)}{\sin(\pi\Delta - \theta)} \tag{10}$$

this imposes a restriction for θ, being that $-\pi\Delta < \theta < \pi\Delta$. Besides, the constant b can be related to both $\mathcal{E}$ and θ as follows

$$b = \frac{1-2\Delta}{2\pi} \cdot \frac{\sin(2\pi\Delta)}{2\cos\pi(\Delta + i\mathcal{E})\cos\pi(\Delta - i\mathcal{E})} = \frac{1-2\Delta}{2\pi} \cdot \frac{2\sin(\pi\Delta+\theta)\sin(\pi\Delta-\theta)}{\sin 2(\pi\Delta)}. \tag{11}$$

Notice that when $\mu = 0$ then $\mathcal{E} = \theta = 0$ and we recover the usual Majorana two-point function and its corresponding expression of b.

This solution can be extended to the non-zero temperature case, obtaining

$$G_c^\beta(\tau) = -b^\Delta \left(\frac{\beta J}{\pi}\sin\frac{\pi\tau}{\beta}\right)^{-2\Delta} e^{2\pi\mathcal{E}\left(\frac{1}{2}-\frac{\tau}{\beta}\right)}, \qquad \beta J \gg 1, \qquad \tau \in [0, \beta]. \tag{12}$$

Since the spectral parameters are unknown, a way to obtain the value of $\mathcal{E}$ is to solve numerically the SD equations and fit it with the expression (12) for large βJ values. The solution is shown in Figure 1.

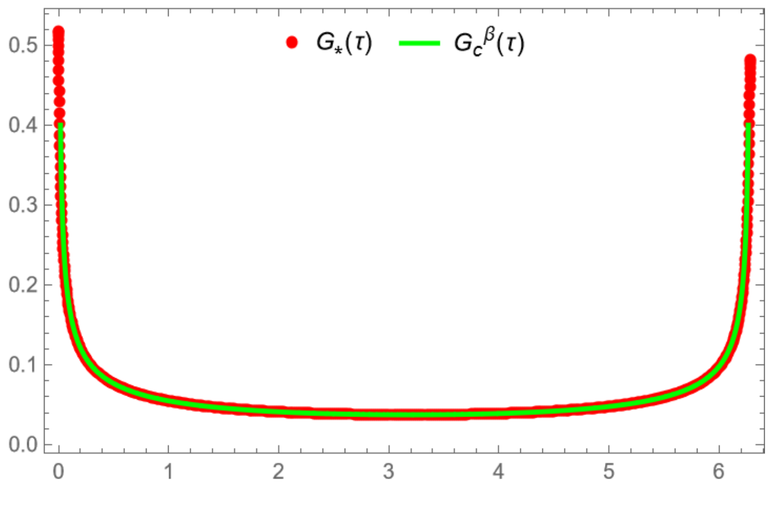

Figure 1: Two-point function solution ($-G(\tau)$) as a function of the imaginary time $\tau \in [0, \beta]$ with $\mu = 1.75$ and $\beta J = 200\pi$. The red doted line is the exact solution of the SD equations and the green continuous line is the conformal one.

A first question to ask is if this conformal solution can be extrapolated to higher energies with the IR parameter $\mathcal{E}$ being a function of the UV one μ.

4 Eigenvalues of the Kernel

As in the Majorana SYK formalism, the diagonalization of the kernel is necessary to know the corresponding scaling dimensions of the operators in the model. Doing so is not a trivial calculation, therefore we will write the steps of obtaining the eigenvalues of $K(\tau_1, \tau_2; \tau_3, \tau_4)$:

i. Given the conformal $SL(2,\mathbb{R})$ generators:

$$\begin{aligned} L_0^{(\tau)} &= -\tau\partial_\tau - \Delta \\ L_{-1}^{(\tau)} &= -\partial_\tau \\ L_1^{(\tau)} &= -\tau^2\partial_\tau - 2\tau\Delta, \end{aligned}$$

the kernel commutes with these conformal generators, namely,

$$\left(L_i^{(1)} + L_i^{(2)}\right) K(\tau_1, \tau_2; \tau_3, \tau_4) = K(\tau_1, \tau_2; \tau_3, \tau_4) \left(L_i^{(3)} + L_i^{(4)}\right).$$

Then, the kernel also commutes with the Casimir:

$$C^{(12)} = \left(L_0^{(1)} + L_0^{(2)}\right)^2 - \frac{1}{2}\left\{L_{-1}^{(1)}L_{-1}^{(2)}, L_1^{(1)}L_{-1}^{(2)}\right\}$$

and thus, both K and $C^{(12)}$ are simultaneously diagonalizable.

ii. The Casimir is diagonalized by conformal three-point functions. In particular, we are interested in correlation functions of the form of 2 complex conjugated fermions of dimension Δ and 1 bosonic operator of dimension h, viz.

$$\langle\psi_j(\tau_1)\psi_j^\dagger(\tau_2)\mathcal{O}_h(\tau_0)\rangle .$$

Depending on the operator $\mathcal{O}_h$, the three-point function can be either symmetric or anti-symmetric under the exchange of fermions (take $\mathcal{O}_h = \psi^\dagger\psi$ or $\mathcal{O}_h = \mathbb{I}$, for instance).

In general, the 3-point function can be written as a sum of symmetric and anti-symmetric parts (CON RESPECTO A QUE?), i.e.

$$\begin{aligned} \frac{1}{N}\langle\psi_j(\tau_1)\psi_j^\dagger(\tau_2)\mathcal{O}_h(\tau_0)\rangle &= f_h^A(\tau_1, \tau_2, \tau_0) + i f_h^S(\tau_1, \tau_2, \tau_0) \\ &= e^{sgn(\tau_{12})\pi\mathcal{E}} b^\Delta \frac{c_h^A sgn(\tau_{12}) + i c_h^S sgn(\tau_{10}) sgn(\tau_{20})}{|J\tau_{12}|^{2\Delta-h}|J\tau_{10}|^h|J\tau_{20}|^h}, \end{aligned}$$

where $c_h^{A/S}$ are determined by solving the eigenvalue equation for the kernel with[4] $k^{A/S}(h) = 1$. CHEQUEAR!!!

iii. Using the three-point functions as eigenfunctions of the Casimir give us the eigenvalues $h(h-1)$, or equivalently

$$C^{(12)}\langle\psi_j(\tau_1)\psi_j^\dagger(\tau_2)\mathcal{O}_h(\tau_0)\rangle = h(h-1)\langle\psi_j(\tau_1)\psi_j^\dagger(\tau_2)\mathcal{O}_h(\tau_0)\rangle ,$$

thus, they are eigenfunctions of the kernel as well.

[4]The values for c_h^A are obtained in the Majorana notes, and the ones for c_h^S are shown in the appendix B.

iv. To find the corresponding eigenvalues of the kernel, separate it as a symmetric and an anti-symmetric operator, depending if it acts on f_h^S or f_h^A, respectively[5].

Starting with the anti-symmetric case K^A, we need to solve

$$\int d\tau_3 \int d\tau_4 K^A(\tau_1,\tau_2,\tau_3,\tau_4) f_h^A(\tau_3,\tau_4,\tau_0) = k^A(h) f_h^A(\tau_1,\tau_2,\tau_0).$$

This eigenvalue equation was already solved in the Maldacena and Stanford paper, so the result is

$$k^A(h,q) = -(q-1)\frac{\Gamma\left(\frac{3}{2}-\frac{1}{q}\right)\Gamma\left(1-\frac{1}{q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)\Gamma\left(\frac{1}{q}\right)}\frac{\Gamma\left(\frac{1}{q}+\frac{h}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{q}-\frac{h}{2}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{q}-\frac{h}{2}\right)}{\Gamma\left(1-\frac{1}{q}+\frac{h}{2}\right)}, \tag{13}$$

which is the result for the Majorana SYK case.

Similarly, the for the symmetric case K^S

$$\int d\tau_3 \int d\tau_4 K^S(\tau_1,\tau_2,\tau_3,\tau_4) f_h^S(\tau_3,\tau_4,\tau_0) = k^S(h) f_S^A(\tau_1,\tau_2,\tau_0),$$

gives

$$k^S(h,q) = \frac{1}{\pi}\frac{\Gamma\left(1-\frac{2}{q}\right)}{\Gamma\left(\frac{2}{q}-1\right)}\Gamma\left(\frac{2}{q}-h\right)\Gamma\left(\frac{2}{q}+h-1\right)\left(\sin\pi h+\sin\left(\frac{2\pi}{q}\right)\right). \tag{14}$$

From both expressions, we can see that there are two types of scaling dimensions (given by solving $k(h)=1$):

$$h_m^A = 1+2m+2\Delta+\epsilon_m^A, \qquad h_m^S = 2m+2\Delta+\epsilon_m^S, \qquad m\geq 0. \tag{15}$$

Now, to write down the form of the primary operators, we see that there is a $U(N)$ symmetry in the effective action (5). This is easier to see if we consider the two-point function as a vector product, i.e. $G\sim\psi^\dagger\psi$ and

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_N\end{pmatrix}.$$

With this symmetry and since, for large value of m, the scaling dimensions have the form

$$h_m^A = 1+2m+2\Delta, \qquad h_m^S = 2m+2\Delta,$$

then, naively

$$\mathcal{O}_{h_m^A} = \psi_j^\dagger\partial_\tau^{2m+1}\psi_j, \qquad \mathcal{O}_{h_m^S} = \psi_j^\dagger\partial_\tau^{2m}\psi_j.$$

[5]The symmetric and anti-symmetric kernels differ by a factor of $(q-1)$, meaning:

$$K^A(\tau_1,\tau_2;\tau_3,\tau_4) = (q-1)K^S(\tau_1,\tau_2;\tau_3,\tau_4) = -J^2(q-1)G(\tau_1,\tau_3)G(\tau_2,\tau_4)G(\tau_3,\tau_4)^{q-2}.$$

Although, the real form should be

$$\mathcal{O}_{h_m^A} = \sum_{i=1}^{N}\sum_{k=0}^{2m+1} d_{mk}^A \partial_\tau^k \psi_j^\dagger(\tau)\partial_\tau^{2m+1-k}\psi_j(\tau), \qquad \mathcal{O}_{h_m^S} = \sum_{i=1}^{N}\sum_{k=0}^{2m} d_{mk}^S \partial_\tau^k \psi_j^\dagger(\tau)\partial_\tau^{2m-k}\psi_j(\tau), \quad (16)$$

where the coefficients $d_{mk}^{A/S}$ are chosen in such a way that the operators are primary.

If we consider the $h_0^A = 2$ and $h_0^S = 0$ modes, we have the following operators:

$$\mathcal{O}_{h_0^A} = \psi_j^\dagger \partial_\tau \psi_j, \qquad \mathcal{O}_{h_0^S} = \psi_j^\dagger \psi_j.$$

We can essentially identify $\mathcal{O}_{h_0^A}$ with the kinetic term in the Hamiltonian of the theory and $\mathcal{O}_{h_0^S}$ with the $U(1)$ conserved charge $\mathcal{Q}$.

Going back to the expressions for the eigenvalues of the kernel, we can see from (13) and (14) that $k^{A/S}(h,q) = k^{A/S}(1-h,q)$. Therefore, it is conventional to use this symmetry to obtain only positive dimensions of the primaries. We thus identify the charge with the $h=1$ mode.

It is interesting to see what are the first scaling dimensions in both the symmetric and anti-symmetric primary operators, to do this we solve numerically $k^{A/S}(h,q) = 1$ and extract from it the desired dimensions, see Figure 2. The first non-integer dimensions are[6]:

$$h_1^A(4) \approx 3.773535618638, \qquad h_2^A(4) \approx 5.679458989211, \qquad h_3^A(4) \approx 7.631970759041.$$
$$h_1^S(4) \approx 2.645744034604, \qquad h_2^S(4) \approx 4.577673896311, \qquad h_3^S(4) \approx 6.552472559916.$$

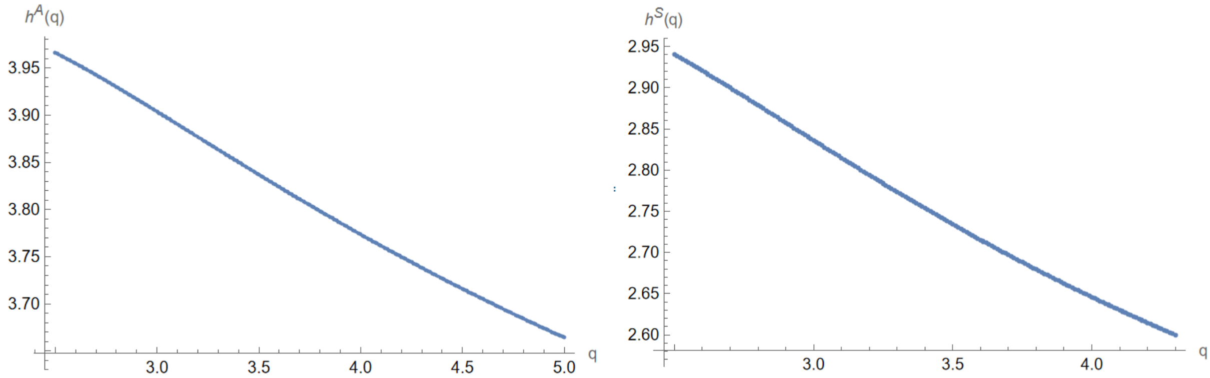

Figure 2: Scaling dimension as a function of q, from here we can extract the first non-integer values of h^A (left) and h^S (right) by solving $k^{A/S}(h,q) = 1$, respectively.

5 cSYK phenomenological approach

In the regime close to the IR (strong coupling or low temperatures), we can consider the cSYK as a CFT plus a perturbation given by the infinite set of irrelevant primaries:

$$I_{cSYK} = I_{CFT} + \sum_h g_h \int d\tau \mathcal{O}_h^{A/S}(\tau), \tag{17}$$

[6]This solutions are for the case $\theta = 0$, that is $\mu = 0$. In general, the scaling dimensions depend on θ since $k^{A/S} = k^{A/S}(h,q,\theta)$. See Appendix D

where the sum runs over all the scaling dimensions $h_0^{A/S}$, $h_1^{A/S}$, $h_2^{A/S},\dots$ and g_h is the perturbative coupling. Following the same calculations as in the Majorana SYK document, we obtain for the two-point function:

$$
\begin{aligned}
G(\tau_{12}) = G_c^0(\tau_{12}) &+ \sum_h g_h \int d\tau_3 \frac{1}{N} \langle \psi_j(\tau_1)\psi_j^\dagger(\tau_2)\mathcal{O}_h(\tau_3)\rangle - \\
&- \frac{1}{2}\sum_{h,h'} g_h g_{h'} \int d\tau_3 \int d\tau_4 \frac{1}{N} \langle \psi_j(\tau_1)\psi_j^\dagger(\tau_2)\mathcal{O}_h(\tau_3)\mathcal{O}_{h'}(\tau_4)\rangle + \cdots
\end{aligned}
\tag{18}
$$

Giving us the zero temperature, up to third order correction[7], $G(\tau)$

$$
G(\tau) = G_c^0(\tau)\left[1 - \sum_h \frac{\alpha_h v_h}{|J\tau|^{h-1}} - \sum_{h,h'} \frac{a_{hh'}\alpha_h\alpha_{h'}}{|J\tau|^{h+h'-2}} - \sum_{h,h',h''} \frac{a_{hh'h''}\alpha_h\alpha_{h'}\alpha_{h''}}{|J\tau|^{h+h'+h''-3}} - \cdots\right], \tag{19}
$$

where the coefficients α_h are unknown (depend on g_h),

$$
v_{h^{A/S}}(\theta) = \frac{1}{1 + (2\Delta - 1)\frac{\sin(\pi h)}{\sin(2\pi\Delta)}} \begin{pmatrix} \frac{\sin(2\theta)}{\sin(2\pi\Delta)}(2\Delta - 1 - \cos(\pi h)) \pm \sqrt{P} \\ 1 + \frac{\sin(2\theta)}{\sin(2\pi\Delta)} + (2\Delta - 1)\frac{\sin(\pi h - 2\theta)}{\sin(2\pi\Delta)} \end{pmatrix},
$$

$$
P = \sin(2\theta)^2\left(1 - \frac{\sin(\pi h)^2}{\sin(2\pi\Delta)^2}\right) + \left(\cos(2\theta) + (2\Delta - 1)\frac{\sin(\pi h)}{\sin(2\pi\Delta)}\right)^2
$$

and

$$
\begin{aligned}
a_{hh'} = -(1 - W_\Sigma(h+h'-1)W_G)^{-1}\Big[&F(h+h'-1)^{-1}\left(F(h)v_h \cdot F(h')v_{h'}\right) + \\
&+ \frac{1}{8}(q-2)W_\Sigma(h+h'-1)\left(q(v_h + \overline{v}_h)\cdot(v_{h'} + \overline{v}_{h'}) - 4\overline{v}_h \cdot \overline{v}_{h'}\right)\Big],
\end{aligned}
$$

with

$$
\begin{aligned}
W_\Sigma(h) &= \frac{\Gamma(2\Delta - 1 + h)\Gamma(2\Delta - h)}{\Gamma(2\Delta)\Gamma(2\Delta - 1)\sin(2\pi\Delta)} \begin{pmatrix} \sin(\pi h + 2\theta) & -\sin(2\pi\Delta) + \sin(2\theta) \\ -\sin(2\pi\Delta) - \sin(2\theta) & \sin(\pi h - 2\theta) \end{pmatrix} \\
W_G &= \begin{pmatrix} q/2 & q/2 - 1 \\ q/2 - 1 & q/2 \end{pmatrix} \\
F(h) &= -i\sqrt{\frac{\Gamma(2\Delta)b}{\Gamma(2 - 2\Delta)}}\Gamma(2 - 2\Delta - h)\begin{pmatrix} e^{i\theta} & 0 \\ 0 & -e^{-i\theta} \end{pmatrix}\begin{pmatrix} i^{2-2\Delta-h} & i^{2\Delta+h-2} \\ i^{2\Delta+h-2} & i^{2-2\Delta-h} \end{pmatrix}\begin{pmatrix} e^{\pi\mathcal{E}} & 0 \\ 0 & -e^{-\pi\mathcal{E}} \end{pmatrix}.
\end{aligned}
$$

The notation $v_h = (v_h^+, v_h^-)^T$, $v_h \cdot v_{h'} = (v_h^+ v_{h'}^+, v_h^- v_{h'}^-)^T$, and $\overline{v}_h = (v_h^-, v_h^+)^T$.

To generalize this result to the finite temperature case, we first rewrite the conformal two point function in the interval $\tau \in [-\beta, \beta]$:

$$
G_c^\beta(\tau) = -\frac{b^\Delta sgn(\tau)}{\left|\frac{\beta J}{\pi}\sin\frac{\pi\tau}{\beta}\right|^{2\Delta}} e^{sgn(\tau)\pi\mathcal{E}} e^{-\frac{2\pi\mathcal{E}}{\beta}\tau}. \tag{20}
$$

[7]This part is taken from Tarnopolsky, et. all. (Excitation spectra of quantum matter without quasiparticles I: Sachdev-Ye-Kitaev models).

As before, the corrections now are integrals in the interval $[0, \beta]$ and the expectation values are the thermal ones, meaning $\langle\ \rangle = \langle\ \rangle_\beta$. Thus,

$$\begin{aligned}\delta_h G(\tau) &= \delta_h G^A(\tau) + sgn(\tau)\delta_h G^S(\tau) \\ &= -G_c^\beta(\tau)\frac{\alpha_h}{(\beta J)^{h-1}}\left[\frac{1}{2}(v_{h^A}^+ + v_{h^A}^-)f_h^A(\tau) + sgn(\tau)\frac{1}{2}(v_{h^S}^+ - v_{h^S}^-)f_h^S(\tau)\right],\end{aligned} \tag{21}$$

where

$$\begin{aligned}f_h^A(\tau) &= \frac{(2\pi)^{h-1}\Gamma(h)^2}{2\sin\frac{\pi h}{2}\Gamma(2h-1)}\left(A_h\left(e^{i\frac{2\pi\tau}{\beta}}\right) + A_h\left(e^{-i\frac{2\pi\tau}{\beta}}\right)\right) \\ f_h^S(\tau) &= \frac{(2\pi)^{h-1}\Gamma(h)^2}{2\cos\frac{\pi h}{2}\Gamma(2h-1)}\left(iA_h\left(e^{i\frac{2\pi\tau}{\beta}}\right) - iA_h\left(e^{-i\frac{2\pi\tau}{\beta}}\right)\right), \\ A_h(u) &= (1-u)^h\mathbf{F}(h,h,1;u),\end{aligned}$$

with $\mathbf{F}$ being the regularized hypergeometric function. We can actually solve for the $h_0^A = 2$ case and obtain the following expressions for $f_0^{A/S}$:

$$f_0^A(\tau) = 2 + \frac{\pi - \frac{2\pi|\tau|}{\beta}}{\tan\frac{\pi|\tau|}{\beta}}, \qquad f_0^S = \frac{\pi}{\tan\frac{\pi|\tau|}{\beta}}.$$

So, up to linear order corrections, the two-point function has the following form (for $\tau \in [0, \beta]$):

$$\begin{aligned}G(\tau) &= G_c^\beta(\tau)\Bigg[1 - \frac{1}{2}\sum_h \frac{\alpha_h}{(\beta J)^{h-1}}\left((v_h^+ + v_h^-)f_h^A(\tau) + (v_h^+ - v_h^-)f_h^S(\tau)\right) - \\ &\qquad - \sum_{h,h'}\frac{a_{h,h'}\alpha_h\alpha'_h}{(\beta J)^{h+h'-2}}f_{h,h'}(\tau) - \cdots\Bigg] \\ &= G_c^\beta(\tau)\Bigg\{1 - \frac{\alpha_0^A}{\beta J}\mathcal{F}_0^A(\tau) - \frac{\alpha_1^S}{(\beta J)^{h_1^S-1}}\left[\frac{(v_{h_1^S}^+ - v_{h_1^S}^-)}{2}f_{h_1^S}^S(\tau) + \frac{(v_{h_1^S}^+ + v_{h_1^S}^-)}{2}f_{h_1^S}^A(\tau)\right] - \\ &\qquad - \frac{a_{00}^A(\alpha_0^A)^2}{(\beta J)^2}f_{00}^A(\tau) - \frac{a_{0,1}^{A,S}\alpha_0^A\alpha_1^S}{(\beta J)^{h_1^S}}f_{0,1}^{A,S}(\tau) - \frac{\alpha_1^A}{(\beta J)^{h_1^A-1}}\left[\frac{(v_{h_1^A}^+ + v_{h_1^A}^-)}{2}f_{h_1^A}^A(\tau) + \right. \\ &\qquad \left. + \frac{(v_{h_1^A}^+ - v_{h_1^A}^-)}{2}f_{h_1^A}^S(\tau)\right] - \frac{a_{000}^A(\alpha_0^A)^3}{(\beta J)^3}f_{000}^A(\tau) - \cdots\Bigg\},\end{aligned} \tag{22}$$

where

$$\mathcal{F}_0^A(\tau) = \frac{(v_{h_0^A}^+ - v_{h_0^A}^-)}{2}f_0^S(\tau) + \frac{(v_{h_0^A}^+ + v_{h_0^A}^-)}{2}f_0^A(\tau)$$

and the first values of h are $h_0^A = 2$, $h_1^S = 2.65$, $h_1^A = 3.77$, the notation $a_{m,n}^{A,S}$ represents the value of $a_{h_m^A,h_n^S}$, $a_{mn}^A = a_{h_m^A h_n^A}$ and so on, similarly for $f_{m,n}^{A,S}(\tau)$. Besides,

$$v_0^A = \begin{pmatrix}1 - \frac{3}{2}\sin(2\theta) \\ 1 + \frac{3}{2}\sin(2\theta)\end{pmatrix}.$$

Note. *It is important to emphasize that when $\mu = 0$, $\alpha_h^S = 0$ (including the $h_0^S = 1$ mode), therefore we get the same expansion of the two-point function as in the Majorana SYK case. In fact, the saddle point solution remains the same in both cases (up to a factor of 2), but the difference lies in the perturbation part: since in the SYK model we only have anti-symmetric perturbations of the saddle solution, while in the cSYK model we also have a symmetric part that contributes as well.*

Note that at $\tau = \frac{\beta}{2}$, the symmetric parts are null, namely $f_h^S(\beta/2) = 0$.

Following the same steps as in the Majorana case, we can obtain the grand-potential expansion $\beta\Phi = -N\log Z$

$$\begin{aligned}\beta\Phi &= -\log\left[\int \mathcal{D}[\psi^\dagger\psi]e^{-I_{cSYK}}\right] \\ &= \beta\Phi_{CFT} + \sum_h g_h \int_0^\beta d\tau\, \langle\mathcal{O}_h(\tau)\rangle_\beta^{conn} - \frac{1}{2}\sum_{h,h'} g_h g_{h'} \int_0^\beta d\tau_1 d\tau_2\, \langle\mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\rangle_\beta^{conn} + \\ &\qquad + \frac{1}{3!}\sum_{h,h',h''} g_h g_{h'} g_{h''} \int_0^\beta d\tau_1 d\tau_2 d\tau_3\, \langle\mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\mathcal{O}_{h''}(\tau_3)\rangle_\beta^{conn} + \dots\end{aligned}$$

where $\frac{\beta\Phi_{CFT}}{N} = \beta E_0 - \mathcal{S}_0 - \beta\mu$, E_0 is the ground energy, μ the chemical potential of the model, and $\mathcal{S}_0$ is the cSYK zero-temperature entropy, given by:

$$\mathcal{S}_0 = 2\pi\mathcal{E}\mathcal{Q} + \int_0^{\frac{1}{2}-\Delta} \frac{2\pi x \sin(2\pi x)}{\cosh(2\pi\mathcal{E}) - \cos(2\pi x)} dx. \tag{23}$$

By CFT arguments, the one-point function $\langle\mathcal{O}_h\rangle_\beta = 0$ for $h \neq 2k$, $k = 1, 2, 3, \dots$ Otherwise,

$$\langle\mathcal{O}_h\rangle_\beta = \frac{N b_h}{(\beta J)^h}.$$

Then, the grand-potential gets the form:

$$\beta\Phi = N\beta E_0 - N\mathcal{S}_0 - \beta\mu N + \beta N\sum_h \frac{g_h b_h}{(\beta J)^h} - \frac{1}{2}\sum_h g_h^2 \int_0^\beta d\tau_1 \int_0^\beta d\tau_2\, \langle\mathcal{O}_h(\tau_1)\mathcal{O}_h(\tau_2)\rangle_\beta + \cdots$$

where the fourth term has only contributions from $h = 2k$, $k \in \mathbb{N}$ and $\langle\mathcal{O}_h(\tau_1)\mathcal{O}_{h'}(\tau_2)\rangle_\beta \propto \delta_{h,h'}$.

The fifth term can be calculated explicitly by introducing a UV cutoff[8] $\varepsilon \sim 1/J$:

$$\begin{aligned}g_h^2 \int_0^\beta d\tau_1 \int_0^\beta d\tau_2\, \langle\mathcal{O}_h(\tau_1)\mathcal{O}_h(\tau_2)\rangle_\beta &= N g_h^2 \beta \int_\varepsilon^{\beta-\varepsilon} d\tau \left(\frac{\pi}{\beta J \sin\frac{\pi\tau}{\beta}}\right)^{2h} \\ &= \frac{N(g_h^2/J^2)(\beta J)}{(h-1/2)(\varepsilon J)^{2h-1}} + \frac{N(g_h^2/J^2)}{(\beta J)^{2h-2}} \frac{\pi^{2h-1/2}\Gamma(\frac{1}{2}-h)}{\Gamma(1-h)},\end{aligned}$$

where the first term represents a correction to the ground energy. Finally, we arrive at

$$\beta\Phi = \beta\Phi_{CFT} + N\sum_h \frac{(g_h/J)b_h}{(\beta J)^{h-1}} - \frac{1}{2}\sum_h \left[\frac{N(g_h^2/J^2)(\beta J)}{(h-1/2)(\varepsilon J)^{2h-1}} + \frac{N(g_h^2/J^2)}{(\beta J)^{2h-2}} \frac{\pi^{2h-1/2}\Gamma(\frac{1}{2}-h)}{\Gamma(1-h)}\right] + \cdots$$

[8]This cutoff is of order $1/J$ because at order $\beta J \sim 1$ the conformal limit stops working.

And using the thermodynamic relation $\epsilon = E/J = -\partial_\beta \log Z/J = \partial_\beta(\beta\Phi/(JN))$, the expression for the energy is:

$$\begin{aligned}\epsilon = \epsilon_0 - \frac{\mu}{J} - \sum_h \frac{(g_h/J) b_h (h-1)}{(\beta J)^h} + \\ - \frac{1}{2}\sum_h \left[\frac{(g_h^2/J^2)}{(h-1/2)(\varepsilon J)^{2h-1}} - \frac{(2h-2)(g_h^2/J^2)}{(\beta J)^{2h-1}} \frac{\pi^{2h-1/2}\Gamma(\frac{1}{2}-h)}{\Gamma(1-h)} \right] + \cdots\end{aligned} \tag{24}$$

Recalling that the sum over h runs over all scaling dimensions, i.e. including h_m^A and h_m^S, we rewrite the previous expression as

$$\begin{aligned}\epsilon = \epsilon_0 - \frac{\mu}{J} + \frac{c_2}{(\beta J)^2} + \frac{c_3}{(\beta J)^3} + \frac{c_4}{(\beta J)^4} + \frac{c_{2h_1^S-1}}{(\beta J)^{2h_1^S-1}} + \frac{c_5}{(\beta J)^5} + \frac{c_{2h_1^S}}{(\beta J)^{2h_1^S}} + \\ + \frac{c_{3h_1^S-2}}{(\beta J)^{3h_1^S-2}} + \frac{c_6}{(\beta J)^6} + \frac{c_{2h_1^A-1}}{(\beta J)^{2h_1^A-1}} + \frac{c_{3h_1^S-1}}{(\beta J)^{3h_1^S-1}} + \frac{c_7}{(\beta J)^7} + \frac{c_{2h_1^A}}{(\beta J)^{2h_1^A}} + \ldots\end{aligned} \tag{25}$$

with $h_1^S = 2.65$ and $h_1^A = 3.77$. One has to be careful when obtaining the coefficients numerically, this because the powers $p = 3h_1^S - 2 \approx 5.95$ and $p = 6$ are very close to each other and some precision may be lost. This also applies for the powers $p = 3h_1^S - 1 \approx 6.95$ and $p = 7$.

Notice that the first non-integer exponent comes from the two point function in the free energy calculation $\langle \mathcal{O}_{h_1^S} \mathcal{O}_{h_1^S} \rangle$. Also, numerically, we can see that terms $1/(\beta J)^p$ with $p = h_0^S,\ h_1^{A/S},\ h_1^{A/S}+1,\ h_1^{A/S}+2,\ h_2^{A/S}$ or h_3^S are absent. Therefore, it is natural to assume that we do not expect terms with powers $p = h_m^{A/S} + n$, with $n = 0, 1, 2, \ldots$ to be present in the energy expansion.

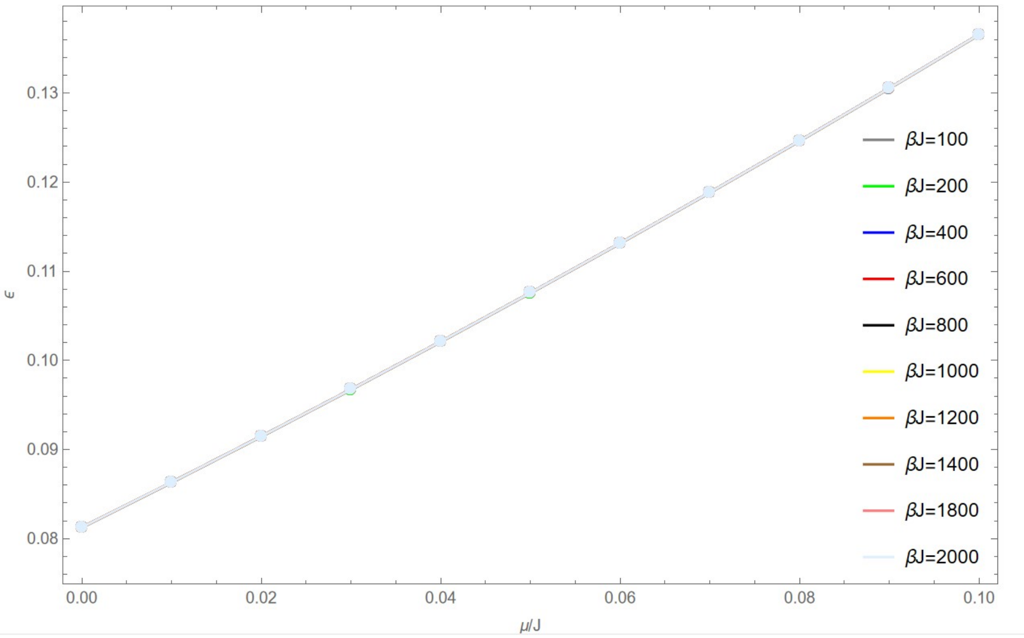

Figure 3: Energy at different temperatures as a function of μ/J. It is noticeable a linear behavior, from which, $\epsilon(\mu = 0) = \epsilon_0$ is twice the value of the Majorana SYK model (since we have twice the number of fields here).

6 The $U(1)$ charge $\mathcal{Q}$

As shown before, a consequence of the non-zero μ in the cSYK model is the presence of a $U(1)$ charge. This quantity represents the so called *particle density*, which in our case is just the number of fermions (there is no space). And it can be calculated by the UV regime of the two-point function, namely:

$$\mathcal{Q} = \frac{1}{2}\left(G(0^+) + G(0^-)\right) = \frac{1}{2}\left(G(0^+) - G(\beta^-)\right). \tag{26}$$

Another expression, derived in Gu, Kitaev, Tarnopolsky, et. all., which depends on the asymmetric parameter θ is:

$$\mathcal{Q} = -\frac{\theta}{\pi} - \left(\frac{1}{2} - \Delta\right)\frac{\sin(2\theta)}{\sin(2\pi\Delta)}, \tag{27}$$

and for the $q = 4$ case:

$$\mathcal{Q} = \frac{1}{4}\left[3 - \tanh(2\pi\mathcal{E})\right] - \frac{1}{\pi}\tan^{-1}\left(e^{2\pi\mathcal{E}}\right). \tag{28}$$

Note. *One can see that both* (26) *and* (27) *give the same observable, but their rhs have different nature. That is, the expression in* (26) *comes from UV behavior of the two-point function, while* (27) *can be obtained from* θ *which is part of the IR solution of* $G(\tau)$*. Therefore, the charge is a quantity that allow us to relate the UV and the IR regimes*[9]*.*

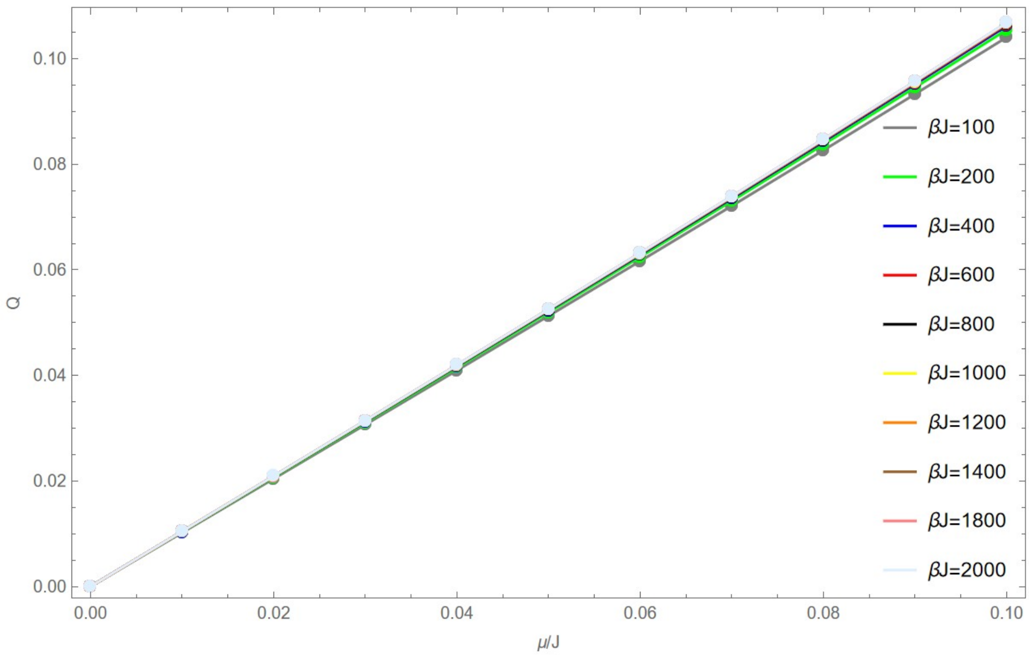

Figure 4: Charge as a function of the chemical potential, from which we can extract the value of the compressibility K, we can see a linear behavior at different temperatures.

Given that we have a non-zero chemical potential, there is an additional thermodynamical quantity that we can calculate: the *Compressibility* K. This quantity relates the particle

[9]This are called Luttinger relations, more on this on Appendix C.

density $\mathcal{Q}$ with the chemical potential μ and is defined as[10]

$$K = \left.\frac{\partial \mathcal{Q}}{\partial \mu}\right|_{\mu \to 0^+} . \tag{29}$$

In our case, and according to Figure 5, the compressibility (as $\beta \to \infty$) has the value of $K = 1.0466359308978943$.

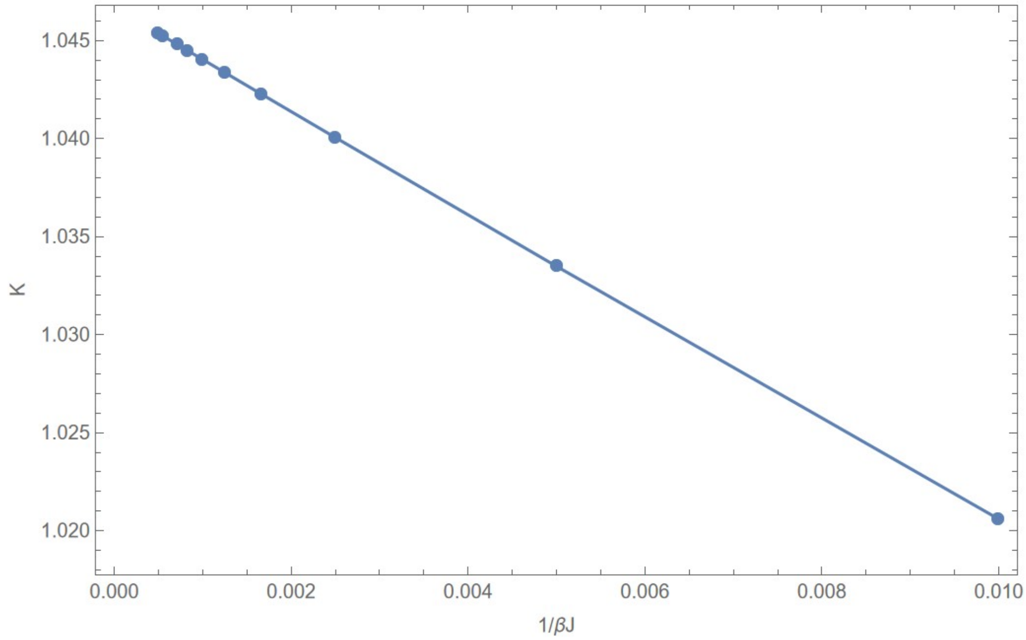

Figure 5: Compressiblity as a function of the temperature, from which we can extract its value from the limit as $T \to 0$, we can see a linear behavior.

A Free two-point function for finite temperature

In Fourier modes, the SD equations can be written as

$$\tilde{G}(i\omega_n) = \frac{1}{i\omega_n + \mu - \tilde{\Sigma}(i\omega_n)}, \tag{30}$$

where in the free case, we just have

$$\tilde{G}_0(i\omega_n) = \frac{1}{i\omega_n + \mu}. \tag{31}$$

To obtain the free two-point function at finite T, we have to calculate the series:

$$G_0^\beta(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \tilde{G}_0(i\omega_n) e^{-i\omega_n \tau} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{e^{-i\omega_n \tau}}{i\omega_n + \mu}. \tag{32}$$

[10]Usually, from statistical mechanics one defines the compressibility as

$$K = \frac{1}{\rho^2} \frac{\partial \rho}{\partial \mu},$$

where ρ is the particle density (in our case, there is no space, so particle density is, essentially, the number of particles).

Using the relation: $\sum_n f(i\omega_n) = \frac{\beta}{2\pi i}\oint_C dz \frac{f(z)}{1+e^{-\beta z}} = -\frac{\beta}{2\pi i}\oint_C dz \frac{f(z)}{1+e^{\beta z}}$, we have

$$\begin{aligned}
\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{-i\omega_n\tau}}{\mu+i\omega_n} &= -\frac{1}{2\pi i}\oint_C dz \frac{1}{(1+e^{\beta z})}\frac{e^{-z\tau}}{(\mu+z)} \\
&= -\frac{1}{2\pi i}\oint_{C'} dz \frac{1}{(1+e^{\beta z})}\frac{e^{-z\tau}}{(\mu+z)} \\
&= -\operatorname*{Res}_{z=-\mu}\frac{e^{-z\tau}}{(1+e^{\beta z})(\mu+z)} \\
&= -\frac{e^{\mu\tau}}{1+e^{-\beta\mu}} \\
&= -\frac{e^{\mu\tau}}{2}\left[1+\left(\frac{\beta\mu}{2}\right)-\frac{1}{3}\left(\frac{\beta\mu}{2}\right)^3+\cdots\right],
\end{aligned}$$

where in the second line we shifted the contour C to C' so that we enclose the pole at $z=-\mu$ and in the last line we use that $\beta\mu \ll 1$ (UV limit).

Similarly,

$$\begin{aligned}
\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{i\omega_n\tau}}{\mu+i\omega_n} &= \frac{1}{2\pi i}\oint_C dz \frac{1}{(1+e^{-\beta z})}\frac{e^{z\tau}}{(\mu+z)} \\
&= \frac{1}{2\pi i}\oint_{C'} dz \frac{1}{(1+e^{-\beta z})}\frac{e^{z\tau}}{(\mu+z)} \\
&= \operatorname*{Res}_{z=-\mu}\frac{e^{z\tau}}{(1+e^{-\beta z})(\mu+z)} \\
&= \frac{e^{-\mu\tau}}{1+e^{\beta\mu}} \\
&= \frac{e^{-\mu\tau}}{2}\left[1-\left(\frac{\beta\mu}{2}\right)+\frac{1}{3}\left(\frac{\beta\mu}{2}\right)^3+\cdots\right].
\end{aligned}$$

Therefore,

$$G_0^\beta(\tau) = \left[-\frac{sgn(\tau)}{2}-\left(\frac{\beta\mu}{4}\right)+\frac{1}{6}\left(\frac{\beta\mu}{2}\right)^3+\cdots\right]e^{\mu\tau}. \tag{33}$$

From this expression, taking the limits as τ approaches zero, we have:

$$G_0^\beta(0^+) = \mathcal{Q}-\frac{1}{2}, \qquad G_0^\beta(0^-) = \mathcal{Q}+\frac{1}{2},$$

and we recover the same behavior as in the zero-T case, if we identify $\mathcal{Q}$ with μ as follows

$$\mathcal{Q} = -\left(\frac{\beta\mu}{4}\right)+\frac{1}{6}\left(\frac{\beta\mu}{2}\right)^3+\cdots \tag{34}$$

B Eigenfunctions of the Symmetric Kernel

C Luttinger relations

Note. *It is conjectured that when gravity is considered, the UV and IR cutoffs of an effective field theory should be related.*

This section is to show explicitly how the UV/IR relation of the charge comes as a phase difference between both asymptotics of the two-point function (as in the usual Luttinger-Ward analysis for Fermi liquids). So, starting from the definition of the charge:

$$
\begin{aligned}
\mathcal{Q}-\frac{1}{2} = G(0^+) &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G(i\omega)e^{i\omega 0^-} \\
&= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\partial_{i\omega}\left(\frac{1}{G(i\omega)}+\Sigma(i\omega)\right)G(i\omega)e^{i\omega 0^-} \\
&= \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\partial_z\left(\frac{1}{G(z)}+\Sigma(z)\right)G(z)e^{z0^-} \\
&= -\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\partial_z\log G(z)e^{z0^-}+\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}G(z)\partial_z\Sigma(z)e^{z0^-}
\end{aligned}
$$

where in the second line we used the SD equations in the form

$$
G(z)=\frac{1}{z+\mu-\Sigma(z)} \implies 1=\partial_z\left(\frac{1}{G(z)}+\Sigma(z)\right).
$$

Clearly, the last expression is logarithmically divergent, so we have to regularize it. Here we show how to solve the first integral, the second one can be referred to Notes on the Complex SYK (Gu, Kitaev, Sachdev, Tarnopolksy):

$$
\begin{aligned}
-\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\partial_z\log G(z)e^{z0^-} &= -\lim_{\eta\to 0}\left[\int_{-i\infty}^{-i\eta}\frac{dz}{2\pi i}\partial_z\log G(z)e^{z0^-}+\int_{i\eta}^{i\infty}\frac{dz}{2\pi i}\partial_z\log G(z)e^{z0^-}\right] \\
&= -\lim_{\eta\to 0}\left[\int_{0+i\eta}^{\infty+i\eta}\frac{dz}{2\pi i}\partial_z\log G(z)e^{z0^-}-\int_{0-i\eta}^{\infty-i\eta}\frac{dz}{2\pi i}\partial_z\log G(z)e^{z0^-}\right] \\
&= -\lim_{\eta\to 0}\left(\int_{0+}^{\infty}\frac{dz}{2\pi i}\partial_z\log\frac{G(z+i\eta)}{G(z-i\eta)}e^{z0^-}\right) \\
&= -\frac{1}{\pi}\lim_{\eta\to 0}\left[\arg G(\infty+i\eta)-\arg G(i\eta)\right].
\end{aligned}
$$

Where in the second line we have used complex analysis with the contours shown in Figure 6 to show that

$$
\begin{aligned}
\oint_{\mathcal{C}_1}dzf(z)=0 &\implies \int_{i\eta}^{i\infty}=\int_{0+i\eta}^{\infty+i\eta}dzf(z) \\
\oint_{\mathcal{C}_2}dzf(z)=0 &\implies \int_{-i\infty}^{-i\eta}=-\int_{0-i\eta}^{\infty-i\eta}dzf(z).
\end{aligned}
$$

And by taking the $\eta\to 0$ limit, we can see that the charge comes from a phase difference between the UV and the IR cut-offs of the Green's function.

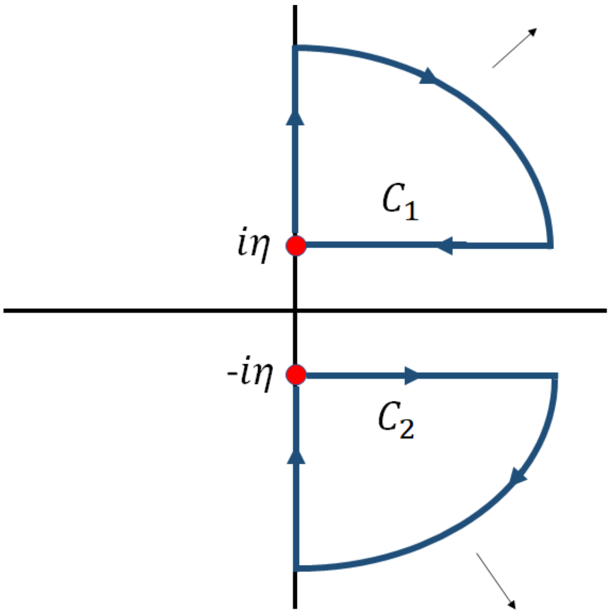

Figure 6: Integral contours that allow us to go from the complex lines to the shifted real lines.

D Kernel eigenvalues for non-zero chemical potential

In the case of non-zero μ, the kernel eigenvalues have the form

$$\begin{aligned} k^A(h,q,\theta) &= \frac{\Gamma\left(\frac{2}{q}-h\right)\Gamma\left(\frac{2}{q}+h-1\right)}{\Gamma\left(\frac{2}{q}+1\right)\Gamma\left(\frac{2}{q}-1\right)}\cdot\left(\frac{2}{q}-1+\frac{\cos(2\theta)\sin(\pi h)}{\sin\frac{2\pi}{q}}-\right.\\ &\left.-\sqrt{\sin(2\theta)^2\left(1-\frac{\sin^2(\pi h)}{\sin^2(2\pi/q)}\right)+\left(\cos(2\theta)+(2\Delta-1)\frac{\sin(\pi h)}{\sin(2\pi/q)}\right)^2}\right)\\ k^S(h,q,\theta) &= \frac{\Gamma\left(\frac{2}{q}-h\right)\Gamma\left(\frac{2}{q}+h-1\right)}{\Gamma\left(\frac{2}{q}+1\right)\Gamma\left(\frac{2}{q}-1\right)}\cdot\left(\frac{2}{q}-1+\frac{\cos(2\theta)\sin(\pi h)}{\sin\frac{2\pi}{q}}+\right.\\ &\left.+\sqrt{\sin(2\theta)^2\left(1-\frac{\sin^2(\pi h)}{\sin^2(2\pi/q)}\right)+\left(\cos(2\theta)+(2\Delta-1)\frac{\sin(\pi h)}{\sin(2\pi/q)}\right)^2}\right). \end{aligned} \tag{35}$$

The integer values of h are protected (independent of θ), therefore $h_0^{A/S}$ are the same regardless of the values of θ that we choose. However, the rest of scaling dimensions depend on the asymmetric parameter, namely, $h_m^{A/S} = h_m^{A/S}(\theta)$ for $m > 0$.

Note. *It is worth mentioning that when $\theta \neq 0$, the action of $K^{A/S}$ on the basis eigenfunctions does not return an anti-symmetric/symmetric eigenfunction again, but instead a mixture of both. In contrast with the $\mu = 0$ case, in which we do have the same parity eigenfunction as a return after the action of the kernel.*

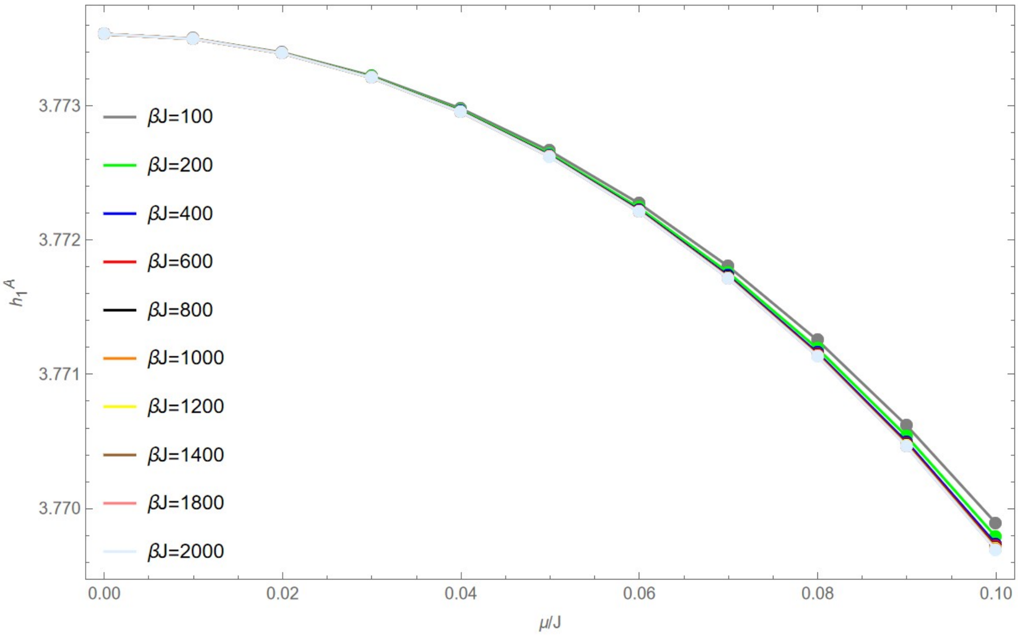

Figure 7: h_1^A vs. μ/J (that depends implicitly on θ). We can see that the scaling dimension reduces as the values of chemical potential become larger.

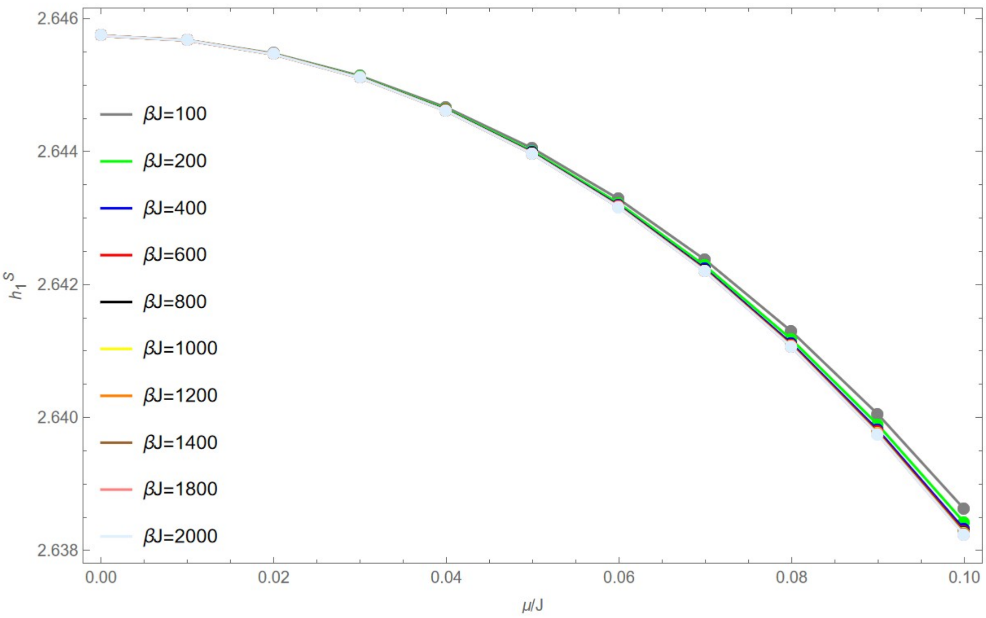

Figure 8: h_1^S vs. μ/J. Again, the scaling dimension reduces as the values of chemical potential become larger at different temperatures.

E Strange Metals and Holographic description

This section is based on Bekenstein-Hawking Entropy and Strange Metals by S. Sachdev.

Strange metals are quantum states without quasiparticles that have a $U(1)$ conserved charge that is continuously variable and it is not spontaneously broken (neither the translational symmetry).

A first description or a holographic duality comes from (23), where

$$\frac{\partial \mathcal{S}_0}{\partial \mathcal{Q}} = 2\pi\mathcal{E}. \tag{36}$$

If we focus on extremal charged black holes (BH) with two-dimensional anti–de Sitter (AdS_2) horizons. For instance, with the Einstein-Maxwell theory of (planar or spherical) charged black holes embedded in asymptotically AdS_{d+2} space, with $d \geq 2$ (the Reissner-Nordström-AdS solution), it was found that

$$\frac{\partial \mathcal{S}_{BH}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}, \tag{37}$$

where $\mathcal{S}_{BH}$ is the Bekenstein-Hawking entropy density of the black hole, $\mathcal{Q}$ is the BH charge density, and $\mathcal{E}$ is the strength of the electric field. This relation comes from the laws of BH thermodynamics (and by using natural units: $\hbar = c = k_B = 1$).

E.1 Charged Black Holes

E.1.1 Planar charged BH

Consider an Einstein-Maxwell theory of gravity with a $U(1)$ gauge field strength $F_{\mu\nu}$:

$$I_{cBH} = \frac{1}{16\pi G_N}\int d^{d+2}x\sqrt{-g}\left[\mathcal{R} + \frac{d(d+1)}{r_{AdS}^2} - \frac{r_{AdS}^2}{g_F^2}F^2\right], \tag{38}$$

where g is the metric, $\mathcal{R}$ is the Ricci scalar, r_{AdS} is the radius of AdS_{d+2}, and g_F is the gauge coupling constant. The solutions of this action for the metric and gauge field leads to:

$$ds^2 = \frac{r^2}{r_{AdS}^2}\left(-f dt^2 + d\vec{x}^2\right) + \frac{r_{AdS}^2}{r^2}\frac{dr^2}{f}, \qquad A = \mu\left(1 - \frac{r_0^{d-1}}{r^{d-1}}\right)dt. \tag{39}$$

with

$$f = 1 + \frac{\Theta^2}{r^{2d}} - \left(r_0^{d+1} + \frac{\Theta^2}{r_0^{d-1}}\right)\frac{1}{r^{d+1}}.$$

The parameters Θ, r_0, and μ are determined by the charge density $\mathcal{Q}$ and the temperature T of the boundary theory in the following way

$$\begin{aligned}
\mu &= \sqrt{\frac{d}{2(d-1)}}\frac{g_F\Theta}{R^2 r_0^{d-1}}, \\
\mathcal{Q} &= \sqrt{2d(d-1)}\frac{\Theta}{8\pi G_N r_{AdS}^d g_F}, \\
T &= \frac{(d+1)r_0}{4\pi r_{AdS}^2}\left(1 - \frac{(d-1)\Theta^2}{(d+1)r_0^{2d}}\right).
\end{aligned}$$

The Bekenstein-Hawking entropy density of this solution (in natural units) is

$$\mathcal{S}_{BH} = \frac{A_{BH}}{4G} = \frac{1}{4G}\left(\frac{r_0}{r_{AdS}}\right)^d. \tag{40}$$

With this in mind, let us focus near the horizon geometry, which at $T = 0$ is at

$$r_h = \left(\frac{(d-1)\Theta^2}{(d+1)}\right)^{1/(2d)}, \tag{41}$$

so, near the horizon let ζ be

$$\frac{1}{\zeta} \equiv r - r_h. \tag{42}$$

Rewriting the metric in terms of ζ, near the horizon ($\zeta \to \infty$) and at $T = 0$:

$$ds^2 = \frac{r_{AdS}^2}{d(d+1)} \frac{(-dt^2 + d\zeta^2)}{\zeta^2} + \frac{r_h^2}{r_{AdS}^2} d\vec{x}^2, \tag{43}$$

and the gauge field becomes

$$A = \frac{\mathcal{E}}{\zeta} dt. \tag{44}$$

The geometry has factorized into $\text{AdS}_2 \times \mathbb{R}^d$, with AdS_2 radius[11]:

$$R_{AdS} = \frac{r_{AdS}}{\sqrt{d(d+1)}}.$$

And the expression of A determines the strength of the AdS_2 electric field in terms of the dimensionless parameter $\mathcal{E}$, which is (in this calculation)

$$\mathcal{E} = \frac{g_F sgn(\mathcal{Q})}{\sqrt{2d(d+1)}}. \tag{45}$$

This is called the equation of state[12] that relates $\mathcal{Q}$ and $\mathcal{E}$, analogous to (27) (using (10) to replace θ with $\mathcal{E}$).

Also, by taking the $T \to 0$ limit in $\mathcal{S}_{BH}$, we have that

$$\mathcal{S}_{BH} = \frac{2\pi g_F |\mathcal{Q}|}{\sqrt{2d(d+1)}} = 2\pi \mathcal{Q}\mathcal{E}.$$

We finally need to show the equivalence of $\mathcal{E}$ from both, the electric field in the AdS picture and the asymetric parameter in the cSYK model. To do this, we need the generalization of the metric and gauge field in a finite temperature case:

$$ds^2 = \frac{R_{AdS}^2}{\zeta^2}\left[-\left(1 - \frac{\zeta^2}{\zeta_0^2}\right) dt^2 + \frac{d\zeta^2}{(1 - \zeta^2/\zeta_0^2)}\right], \qquad A = \mathcal{E}\left(\frac{1}{\zeta} - \frac{1}{\zeta_0}\right) dt, \tag{46}$$

where $\zeta_0 = \frac{1}{2\pi T}$. Hence, the action of a matter field Ψ (fermionic spinor), with charge $q = 1$ moving in this background is:

$$I_{matter} = i \int d^2x \sqrt{-g} \left(\bar{\Psi} \not{D}\Psi - m\bar{\Psi}\Psi\right),$$

with $\not{D} = \gamma^\mu \mathrm{D}_\mu$, γ^μ are the gamma matrices and D_μ is the covariant derivative with charge $q = 1$. The thermal AdS_2 correlator of this fermions in an electric field background has the form:

[11] This is a metric of a planar charged BH, the results for a spherical one are given below.

[12] Again, this relation is not the same as for the spherical BH one. But, the more fundamental relation $\partial\mathcal{S}/\partial\mathcal{Q} = 2\pi\mathcal{E}$ is the same in both cases (and in cSYK as well).

$$G(\omega) = -\frac{iCe^{-i\theta}}{(2\pi T)^{1-2\Delta}} \frac{\Gamma\left(\Delta - \frac{i(\omega-\omega_S)}{2\pi T}\right)}{\Gamma\left(1-\Delta - \frac{i(\omega-\omega_S)}{2\pi T}\right)},$$

where $\Delta = 1/4$, C is real and positive and $\omega_S = T\frac{\partial \mathcal{S}}{\partial \mathcal{Q}}$. This correlator has the same form as in the cSYK model. This computation shows that $2\pi\mathcal{E} = \partial\mathcal{S}/\partial\mathcal{Q}$, the same as in the cSYK case. The scaling dimension Δ is related to the AdS_2 spinor mass m:

$$\Delta = \frac{1}{2} - \sqrt{m^2 R_{AdS}^2 - \mathcal{E}^2}.$$

E.1.2 Spherical charged BH

Here, we summarize the results (as in the previous case) of the spherical solution of a charged BH. In this case, we consider a solution of (38) with metric:

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_d^2, \qquad V(r) = 1 + \frac{r^2}{r_{AdS}^2} + \frac{\Theta^2}{r^{2d-2}} - \frac{M}{r^{d-1}}, \tag{47}$$

where M is determined by the zero of $V(r)$ at r_0. Following the same steps for $T = 0$ in the planar BH case, one finds

$$\Theta^2 = \frac{r_0^{2d-2}\left[(d-1)r_{AdS}^2 + (d+1)r_0^2\right]}{(d-1)r_{AdS}^2},$$

near the horizon we define:

$$r - r_0 = \frac{R_{AdS}^2}{\zeta},$$

with

$$R_{AdS} = \frac{r_{AdS}}{\sqrt{d(d+1) + (d-1)^2 r_{AdS}^2/r_0^2}}$$

so that near the horizon, the metric becomes $\mathrm{AdS}_2 \times \mathbb{S}^d$

$$ds^2 = R_{AdS}^2\left(\frac{-dt^2 + d\zeta^2}{\zeta^2}\right) + r_0^2 d\Omega_d^2 \tag{48}$$

and in the gauge field sector, we have

$$\mathcal{Q} = \frac{r_0^{d-1}\sqrt{2d\left[(d-1)R_{AdS}^2 + (d+1)r_0^2\right]}}{8\pi G_N g_F},$$
$$\mathcal{E} = \frac{g_F r_0 \sqrt{2d\left[(d-1)R_{AdS}^2 + (d+1)r_0^2\right]}}{2\left[(d-1)^2 r_{AdS}^2 + d(d+1)r_0^2\right]},$$

the equation of state, and the explicit form of the entropy density, can be obtained eliminating r_0 from both equations, so that the fundamental equation

$$\frac{\partial \mathcal{S}_{BH}}{\partial \mathcal{Q}} = \frac{\partial \mathcal{S}_{BH}/\partial r_0}{\partial \mathcal{Q}/\partial r_0} = 2\pi\mathcal{E}$$

is satisfied, as expected.

Note. *It is worth to mention that for charged black holes, as the corresponding mass approaches the charge (near-extremal Reissner-Nordstrom BH) a throat develops whose geometry is similar to that of* $AdS_2 \times S^2$*. And the* AdS_2 *part for this system is said to be dual to a nearly conformal system with no spatial dimension, namely, the cSYK model in the IR regime.*

Such throat is unique for charged BH, there is no such analogue for Schwarzschild ones.