Conformal Perturbation Theory for O'Brien Fendley model

(Dated: October 16, 2025)

I. CPT ON ISING CFT

Here, we consider the lattice O'Brien Fendley model (OF) [1], which is described by an Ising Hamiltonian at the critical point and a next-to-nearest neighboring term that describes quartic Majorana fermionic interactions. The latter flows to the irrelevant operator $T\bar{T}$ (with respect to Ising CFT). The Hamiltonian reads:

$$H_{OF} = \underbrace{2\alpha\left(-\sum_{n=0}^{N-1} X_n X_{n+1} - \sum_{n=0}^{N} Z_n\right)}_{\text{critical Ising}} + \underbrace{\lambda \sum_{n=0}^{N-2} \left(Z_n X_{n+1} X_{n+2} + X_n X_{n+1} Z_{n+2}\right)}_{\text{In the IR, flows to } T\bar{T}}. \tag{1}$$

The second term is similar to the one described by the Interacting Majorana chain model [2], which also flows to $T\bar{T}$ of the $M_{(3,4)}$ minimal model.

The aim is to use Conformal Perturbation Theory (CPT) on Ising-CFT to describe our UV lattice model in (1). Therefore, we move away of the CFT by adding the operators: ε, $T\bar{T}$, and $T_4 + \bar{T}_4$ as follows

$$H_{IR} = H_{CFT} + g_\varepsilon \int d\theta\ \varepsilon + \frac{g_{T\bar{T}}}{R^3} \int d\theta\ T\bar{T} + \frac{g_{T_4}}{R^3} \int d\theta\ (T_4 + \bar{T}_4) + \dots \tag{2}$$

If, after the perturbation, we are still in the Ising universality class, then g_ε has to vanish. This is what we expect in the OF model, since we are perturbing it by an irrelevant deformation that flows to $T\bar{T}$.

Thus, taking $R = Na$, where $a-$ lattice spacing and N number of sites, we can use (2) to describe (1) in an Ising perturbative region, i.e. for $|\lambda| \ll 1$. So, starting from (2), we have that

$$E_{IR}^{\mathcal{O}} = E_0 + \frac{\nu}{N}\left[\Delta_{\mathcal{O}} + g_\varepsilon N \langle\mathcal{O}|\,\varepsilon\,|\mathcal{O}\rangle + \frac{g_{T\bar{T}}}{N^2} \langle\mathcal{O}|\,T\bar{T}\,|\mathcal{O}\rangle + \frac{g_{T_4}}{N^2} \langle\mathcal{O}|\,T_4 + \bar{T}_4\,|\mathcal{O}\rangle + O(N^{-3})\right] \tag{3}$$

where ν is the speed of light that depends on the UV model and we have used that the operators are rotation invariant. Therefore, by using ratios, we can match the IR spectrum with the one given by the OF model. That is,

$$\frac{E_{IR}^{\mathcal{O}} - E_{IR}^{\mathbb{I}}}{E_{IR}^{\partial\sigma} - E_{IR}^{\sigma}} = \frac{E_{OF}^{(s,p,n)} - E_{OF}^{(0,1,0)}}{E_{OF}^{(1,-1,0)} - E_{OF}^{(0,-1,0)}}. \tag{4}$$

with $E_{OF}^{(s,p,n)}$ being the $n-$th energy with spin s, and Ising $\mathbb{Z}_2$ parity p. By fixing $\alpha = 1$ in (1) and performing the previous matching for different energy gaps, we can obtain g_ε, $g_{T\bar{T}}$ and g_{T_4} as a function of λ, as shown in Figure 1. We can see that $g_\varepsilon \simeq 0$, as expected, since we are in Ising universality class. Additionally, both $g_{T\bar{T}}$ and g_{T_4} go from positive to negative values as λ increases from negative to positive. Notice that, at $\lambda = 0$ we have $g_{T\bar{T}} = 0$ which is expected from the Ising model, and, in particular, that at $\lambda \sim -0,18$, we get $g_{T_4} = 0$. The latter result can help us to have a cleaner extrapolation from the lattice NNN term to the $T\bar{T}$ operator in the IR.

An important question to ask is: up to what point could we trust CPT? Checking other scaling dimensions at the parameters obtained from (4), we see that $\lambda \in [-0.5, 0.175]$ using $N = 22$, where we obtain a 10^{-3} error.

II. CPT ON TRI-CRITICAL ISING CFT

Interestingly, the OF model has another critical point in the thermodynamic limit (at $\lambda \sim 0.856$) which belongs to Tri-Critical Ising universality class, namely the $M_{(4,5)}$ minimal model. So, we can perform the same calculations but now using $M_{(4,5)}$ as our background, and do CPT with its corresponding operators. So, we get

$$H_{IR}^{TCI} = H_{CFT} + g_\varepsilon R^{0.8} \int d\theta\ \varepsilon + g_{\varepsilon'} R^{-1.2} \int d\theta\ \varepsilon' + g_{\varepsilon''} R^{-2} \int d\theta\ \varepsilon'' + \dots \tag{5}$$

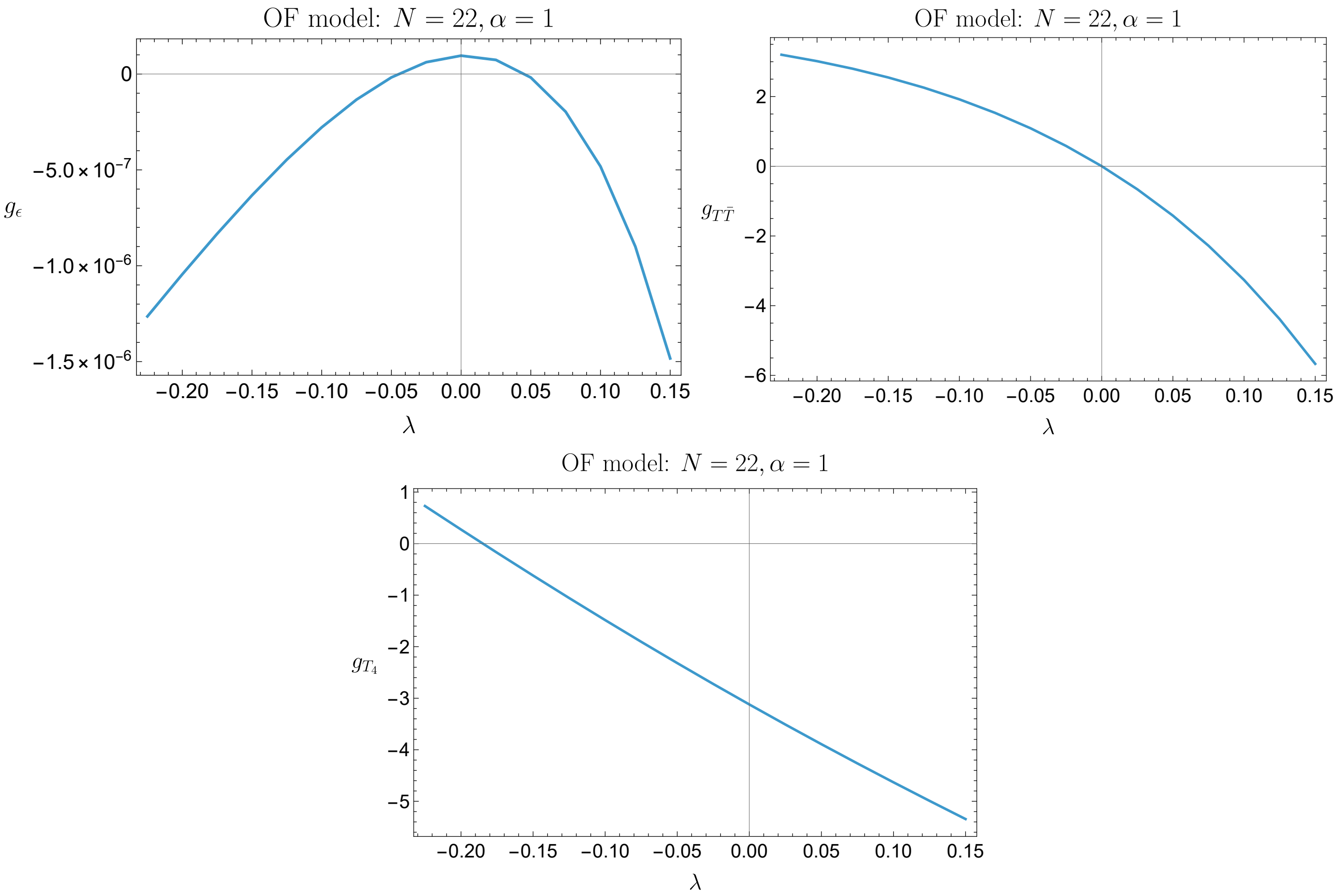

FIG. 1: IR parameters as a function of λ. We see that $g_\varepsilon \simeq 0$, as expected, and both $g_{T\bar{T}}$, g_{T_4} go from positive to negative values.

from which, we write

$$E_{IR}^{\mathcal{O}} = E_0 + \frac{\nu}{N}\left[\Delta_{\mathcal{O}} + g_\varepsilon N^{1.8}\langle\mathcal{O}|\,\varepsilon\,|\mathcal{O}\rangle + g_{\varepsilon'}N^{0.8}\langle\mathcal{O}|\,\varepsilon'\,|\mathcal{O}\rangle + \frac{g_{\varepsilon''}}{N}\langle\mathcal{O}|\,\varepsilon''\,|\mathcal{O}\rangle + O(N^{-2})\right]. \tag{6}$$

Hence, using the ratios (4), we obtain a solution for g_ε, $g_{\varepsilon'}$, $g_{\varepsilon''}$ as a function of λ, as shown in Figure 2. We can see that $g_\varepsilon \simeq 0$, as expected. However, $g_{\varepsilon'} \neq 0$ for small $\delta\lambda(= \lambda - 0.856)$, this means that the CFT flows to a different theory, and this point represents an unstable critical point (which is in accordance with the phase diagram presented in [1]). In fact, perturbing $M_{(4,5)}$ with ε' operator give us a flow to $M_{(3,4)}$ (our Ising criticality[1] located at $\lambda \sim 0$). Finally, checking the other scaling dimensions at the parameters obtained from (4), we see that when $\lambda \in [0.8, 0.9]$, where we obtain a 10^{-2} error for $N = 20$, CPT is reliable.

[1] E. O'Brien and P. Fendley, "Lattice supersymmetry and order-disorder coexistence in the tricritical ising model," *Phys. Rev. Lett.*, vol. 120, p. 206403, May 2018.
[2] A. Rahmani, X. Zhu, M. Franz, and I. Affleck, "Phase diagram of the interacting majorana chain model," *Phys. Rev. B*, vol. 92, p. 235123, Dec 2015.

[1] This operator would correspond to ϕ^4 in the Ginzburgh-Landau description.

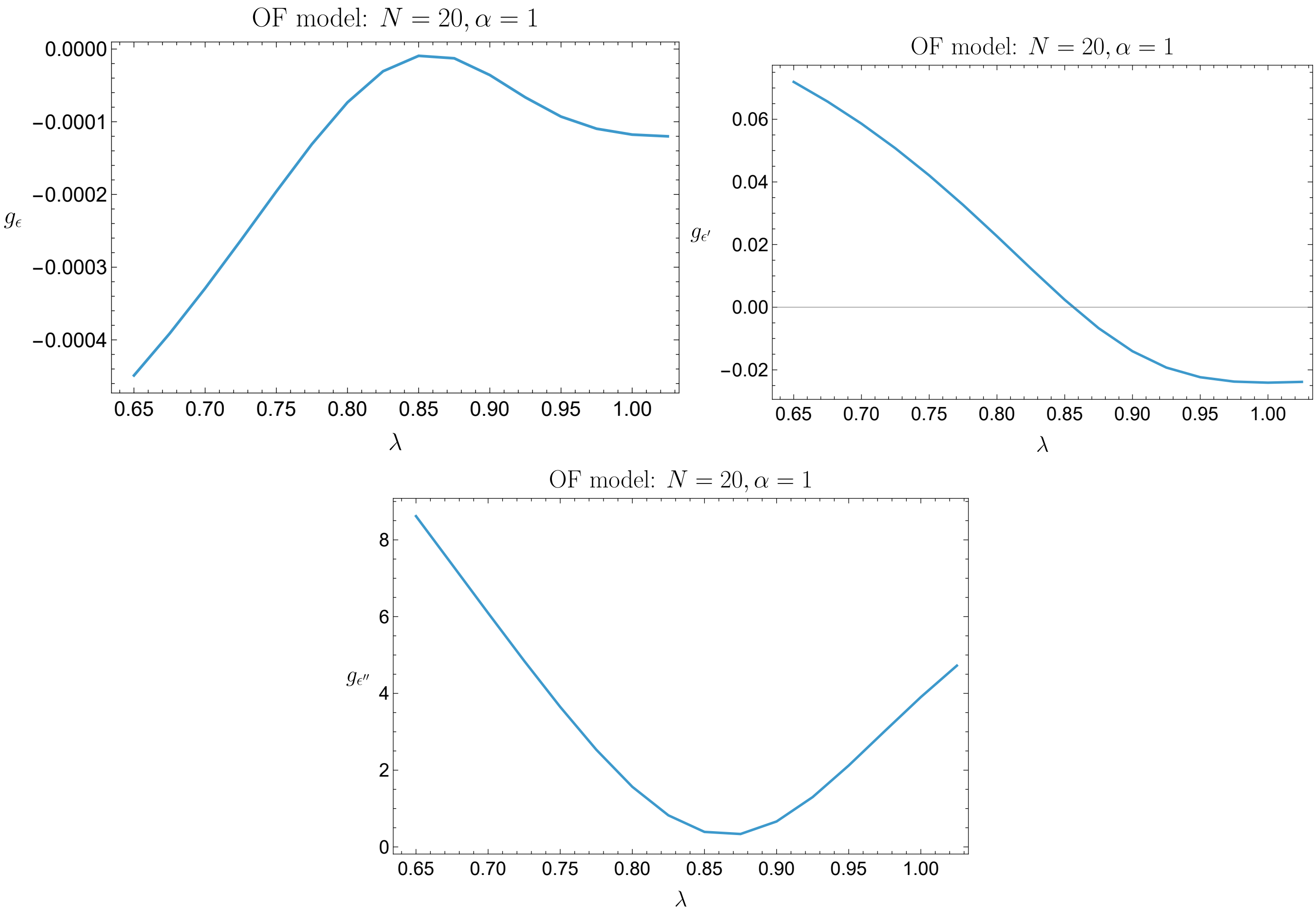

FIG. 2: Tri-Critical Ising parameters as a function of λ for the OF model. We see that $g_\varepsilon, g_{\varepsilon'} \simeq 0$ close to $\lambda = 0.856$.