Conformal Field Theory Notes

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Abstract: In these notes we present a brief introduction of CFTs, following mostly the *yellow book*, also known as Conformal Field Theory by David Sénéchal, Philippe Di Francesco [1], and Pierre Mathieu. We will also follow notes from David Tong, and Qualls[2]

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1 Why CFTs?

Conformal field theories have been one of the most studied and talked about subjects in the recent history of Physics, so what is so special about them? In the following section we will try to give some motivation from different perspectives. Although the course will be given from the point of view of high energy theory, we hope to make statements that are general enough to also please the condensed matter folks.

1.1 From first principles

A way in which physics has become more like mathematics in the recent decades has been the thirst for generality. This is embodied in the many no-go theorems developed over the years. With Einstein's special relativity, we learnt that Lorentz invariance is a symmetry of our universe.In addition to this, we also have translational invariance in spacetime, and GR promoted this global symmetry to a local one, and added diffeomorphism invariance. As gauge theories developed by Weyl and others in the 20s and 30s, and with the proliferation of quantum field theories, and symmetries as a way to organize the particle zoo, symmetries became a powerful tool to organize the spectrum of theories. Now that we discovered that the "symmetries" of the universe formed a much bigger group than initially imagined, the reductionist approach to find a Grand Unified Theory was initiated by Salam, Weinberg, Georgi, Glashow, and others [3][4][5][6]. The question of generality then arose: what kind of symmetries, even in principle, could we have in our universe? The answer did not take long to come from the celebrated Coleman-Mandula theorem [7][1] which assuming: 1. Non-trivial scattering, 2. Massive particles, 3. Finite number of mass states, and 4. Analyticity of the S-Matrix, they were able to prove the wonderfully general result that the only way to

[1]A good discussion on this theorem is in Weinberg Vol 3 Ch24 Appendix B

not violate these assumptions is to have a direct product of Poincare x Internal symmetry group. This completely shut down any hope of some spacetime-gauge unification, but at the same time very much focused the research directions and saved countless vain efforts of unification. Nonetheless, and to the point of these classes, there were two notable loopholes to this theorem. Firstly, the Coleman-Mandula theorem considers only bosonic symmetries. By including fermionic spacetime generators, one arrives at the ubiquitous supersymmetry. On the other hand, one can relax the condition of massive states, and, as shown by Mack and Salam (and Weinberg)[8][9] the most general group that preserves the momentum squared is the conformal group, and gives constraints on what kinds of symmetry breaking are possible (namely, they show internal symmetry breaking is the only possibility). Thus, we conclude the first principles section with the following answer: **we consider conformal symmetry because it is the biggest allowed bosonic extension of Poincare**.

1.2 What is conformal anyway

This will be a brief section as the answer to this question will be covered in excruciating detail throughout the course, but in simple terms, conformal maps are those that locally preserve angles. In 4D, we can clearly see that all re-scalings of coordinates ($x \to \lambda x$) will

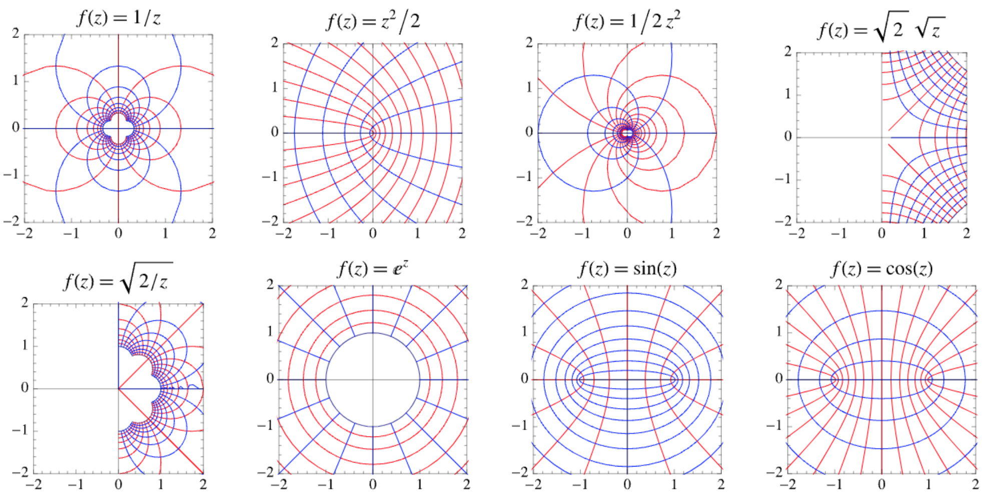


Figure 1. Caption

keep angles the same. Similarly, translations, and rotations will also preserve angles. It turns out (as we will see) that boosts, and "special conformal transformations" also preserve angles. The latter can be seen as a composition of an inversion followed by a translation followed by another inversion. The rule of thumb is that a scale invariant theory will often also be conformally invariant, although this needs not always be the case [10]. In 2d, under certain assumptions, it can be shown that the scale invariance is enhanced to conformal invariance[11]. On the other hand, as a byproduct of dilations being a subgroup of the conformal group, scale invariance is implied by conformal invariance. For more information you can check [12][13]. This means that many theories that we know and love posses this symmetry, as an example Maxwell's equations in the vacuum are scale invariant, although this gets spoiled at the quantum level due to quantum corrections. Similarly, massless free scalars $\mathcal{L} = \partial_\mu\phi\partial^\mu\phi$ also have no scale dependence. In general, one may consider a theory scaleless if there are no dimensionful parameters in the theory. An example of this might be $\lambda\phi^4$ theory in 4d. Nonetheless, the quantum corrections spoil this invariance, although as we raise the cutoff to infinity, the theory flows to the free Lagrangian ($\lambda = 0$). RG flows will key in understanding the importance of Conformal Field theories as we shall see. Other members of this select club include what Witten calls "the simplest theory in the world" a.k.a. $\mathcal{N} = 4$ Super Yang Mills. We also have other examples such as Seiberg-Witten theories, or many other theories at fixed points.

1.3 Stat mech and RG summary[2]

Consider a $\lambda\phi^4$ theory with an additional $\mathbf{Z}_2$ symmetry, meaning there are no odd terms in the lagrangian. We consider this theory on a lattice of spacing a, meaning that there will be a UV cutoff proportional to $1/a$. As usual in the effective field theory (EFT) approach we include all terms that are compatible with our symmetries, giving rise to an action such as

$$S = \int d^dx\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \lambda\phi^4 + g_1\phi^6 + g_2\Box\phi\Box\phi + ... \tag{1.1}$$

Now we consider the space of all coupling constants, and consider how the costants flow as we coarse grain (change the cutoff). We may want to do this if we are interested in the long range physics, so we can average over a higher lattice spacing and consider the resulting theory. As you will have seen in statistical mechanics, averaging over some lattice sites changes the value of the coupling between neighbours in e.g. the Ising model. This is what we mean by flowing; we are going from a theory with a coupling J to one with a coupling J' by coarse graining and looking at the bigger picture. We now turn our attention to the low energy physics that we care about, say below an energy scale of $\Lambda' = \frac{1}{L}$, allowing us to write $\Lambda' = \frac{\Lambda}{\gamma}$ for some γ. Then we can construct our fourier modes as such:

$$\phi_\mathbf{k} = \phi_\mathbf{k}^- + \phi_\mathbf{k}^+$$

where $\phi_\mathbf{k}^-$describe the long-wavelength fluctuations

$$\phi_\mathbf{k}^- = \begin{cases} \phi_\mathbf{k} & k < \Lambda' \\ 0 & k > \Lambda' \end{cases}$$

and $\phi_\mathbf{k}^+$describe the short-wavelength fluctuations that we don't care about

$$\phi_\mathbf{k}^+ = \begin{cases} \phi_\mathbf{k} & \Lambda' < k < \Lambda \\ 0 & \text{otherwise} \end{cases}$$

we now consider integrating out the UV modes, which effectively amounts to putting the the UV modes on-shell at first approximation. We additionally decompose the free energy as

$$F\left[\phi_\mathbf{k}\right] = F_0\left[\phi_\mathbf{k}^-\right] + F_0\left[\phi_\mathbf{k}^+\right] + F_I\left[\phi_\mathbf{k}^-, \phi_\mathbf{k}^+\right] \tag{1.2}$$

[2]For a more detailed discussion, see [14], which this subsection takes as a strong inspiration

so that

$$Z = \int \prod_{k<\Lambda} d\phi_{\mathbf{k}} e^{-F} = \int \prod_{k<\Lambda'} d\phi_{\mathbf{k}}^{-} e^{-F_0[\phi_{\mathbf{k}}^{-}]} \int \prod_{\Lambda'<k<\Lambda} d\phi_{\mathbf{k}}^{+} e^{-F_0[\phi_{\mathbf{k}}^{+}]} e^{-F_I[\phi_{\mathbf{k}}^{-},\phi_{\mathbf{k}}^{+}]}. \tag{1.3}$$

Since we started with a Lagrangian with all possible operators, all that integrating fields out can do is change the coefficients of some of these operators, as the Lagrangian to start with was the most general we could have. Thus, we can write

$$Z = \int \mathcal{D}\phi^{-} e^{-S'[\phi^{-}]} \tag{1.4}$$

where S' is the effective action with different coefficients. This is referred to as Wilsonian renormalisation and lies at the heart of the current understanding of quantum field theory. It is a profound realisation that we can describe a theory at two different energy scales by simply tuning the coefficients of the same theory. Nonetheless, there is one more subtlety to address. The two theories are not equivalent, as the regime of validity is different for both of them. This can be remedied by changing $k' = \gamma k$ so that k' goes up to the original cutoff Λ, and the space rescaling is $x' = \frac{x}{\gamma}$, which is equivalent to the aforementioned coarse graining. As a final step we should canonically normalise fields, should the RG lead to a different kinetic term. Then, shifting $\gamma \in [0, \infty)$ away from 0 traces flows in parameter space, governed by a differential equation known as the beta function. Now the question arises, what happens when we do this procedure *ad infinitum*? The two possibilities we have are either converging on a set of values for the couplings, or being pushed to infinity in parameter space. The former is referred to as fixed points for obvious reasons. Since the theory parameters do not change after the renormalisation procedure, that means that the new length scale will also not change. This in turn implies that this scale must either be zero, or infinity. In Statistical field theory, the scale is often referred to as the correlation length, and it is the scale at which two systems might affect each other (e.g. how far can a spin up affect some other particle with spin down). The hallmark of phase transitions in statistical mechanics is that the correlation length diverges as we approach critical points, and since the correlation length blows up, which gives the average size of "spin bubbles", this rids the system of a typical length scale. Without such restriction, we can have spin bubbles of all sizes as there is no "length/energy". Sound familiar? As a final note, we review the kinds of interactions we may have. These are coined "relevant", "marginal" and "irrelevant" corrections. Relevant corrections are those that change the IR physics, in other words their importance grows as we go to the IR (think of a mass term, if you are in the TeV scale, it is not very relevant whether the rest mass of a particle is that of an electron, or that of a proton, since the kinetic energy will dominate, on the other hand, if we go into the deep IR, the mass will be the most important term). Irrelevant corrections are those that are important in the UV physics, but not in the IR (think many higher order interactions). Finally, marginal interactions happen when the fixed point is a fixed manifold, and the interactions move you through it. One may guess the category of different operators via engineering dimensions of operators, though operators may gain anomalous dimensions through renormalisation, meaning that many seemingly marginal operators are

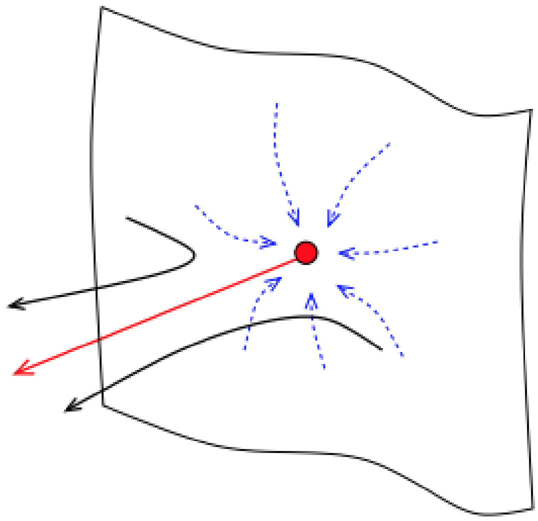


Figure 2. Image from [14]. The blue lines denote irrelevant deformations, while the red ones are relevant ones moving us away from the critical surface.

not so. The reason why universality classes appear is because the vast majority of operators are irrelevant, and so there are comparatively fewer theories that one may flow to in the IR.

1.4 String theory, duh [3]

Conformal invariance is one of the most important aspects of string theory, as we will see, 2D CFTs are one of the most studied kinds of CFTs and they are so for many good reasons, but one of them is their role in string theory. One of the first equations introduced in string theory is the Polyakov action

$$I_1[x,g] = -\frac{1}{2}T\int d^2\xi\sqrt{-g}g^{ab}\partial_a x^\mu \partial_b x_\mu \tag{1.5}$$

which gives the action of a string in a d-dimensional spacetime. It is straightforward to show (**DO IT**) that it is invariant under rescalings of the metric $g_{ab} \to g_{ab}\Omega$. Similarly, it also has reparametrisation invariance, which gives us two further gauge transformations. in particular: 1. Reparametrisation invariance:

$$\xi'^a = f^a(\xi) \quad \to \quad x'^\mu\left(\xi'\right) = x^\mu(\xi), \quad g'_{ab}\left(\xi'\right) = \frac{\partial\xi^c}{\partial\xi'^a}\frac{\partial\xi^d}{\partial\xi'^b}g_{cd}(\xi)$$

2. Weyl invariance:

$$x'^\mu(\xi) = x^\mu(\xi), \quad g'_{ab}(\xi) = \sigma(\xi)g_{ab}$$

[3]this section is partly inspired from [15]

which is the same number of components as the metric in 2d, g_{ab}, and thus, we can completely fix it to be anything you want. If you are not a masochist, this will be flat space metric. It is thus thanks to conformal invariance, that we can have a "nice" string theory. The Polyakov action can be related to the Nambu-Goto action, which has interpretation of a Lagrangian that describes the area swept out by a string, more known as the "worldsheet". The equations of motion then minimise the area, in a similar fashion to how we may expect a particle to follow geodesics. Of course, the love story between string theory and conformal field theory does not stop here. Bosonic string theory is a toy model and the more serious attempts to describe the world come in the form of superstring theory, where the conformal symmetry gets promoted to superconformal invariance as the worldsheet of the string includes fermionic coordinates[4]. Another area where conformal field theory features prominently is the now legendary AdS/CFT correspondence. As we will see, the isometry group of AdS spacetime is the same as the conformal group, but we hope to say more about this in later stages of the course. The selling point of the correspondence, as well as finding a gravity theory (IIB superstring theory on $AdS_5 \times S^5$) that has a dual without gravity ($\mathcal{N} = 4$ Super Yang-Mills), is that the correspondance is also between weakly and strongly couple theories. Needless to say, this is big as we know just as much about strongly coupled theories as one might know about what the next lottery winning number will be.

[4]There are two formalisms to achieve superstring theory, one is worldsheet supersymmetry and the other one is spacetime supersymmetry. These two approaches seem to be equivalent, but no one has a proof of this, they go by the names of Ramond-Neveu-Schwartz (RNS) formalism, and the Green-Schwartz (GS) formalism, which you can read more about in the Becker-Becker-Schwartz book.

2 Conformal Field Theory (CFT)

CFTs can be thought of as the endpoints of RG flows: In the UV, there are no energy scales because all energy parameters are irrelevant; and in the IR, there are no energy scales because all massive modes are not excited. So, what is a correct definition of a CFT? There are different answers depending on the reference we look at, but, in general one can define it to be a Quantum Field Theory with the property that:

$$W[g,J] = W[\Omega^2 g, \Omega^{\Delta-d} J], \tag{2.1}$$

where W is the generating functional, g is the background metric, J is a current, and Ω is a (rescaling) function[5]. This can be translated into the following equation of correlation functions:

$$\boxed{\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{\Omega^2 g} = \Omega(x_1)^{-\Delta_1}\cdots\Omega(x_n)^{-\Delta_n}\,\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_g}. \tag{2.2}$$

This is known as a **Weyl transformation**. A subcase of this is the (conformal) transformation, where $x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu$, where

$$d\tilde{x}^\mu d\tilde{x}^\nu = \Omega(x)^2 dx^\mu dx^\nu.$$

There is a fundamental distinction that we want to stress: A Weyl transformation is not a conformal transformation, but they give the same results. The reason is that a conformal transformation is an effect of a Weyl transformation. In other words, we have two points of view:

1. Keep the points where they are, but change the way of measuring distances (the metric).
2. Keep the metric intact and change the coordinates.

Both cases are equivalent, but not the same!

Focusing on conformal transformations, we can rewrite our equation (2.2) as:

$$\langle \mathcal{O}_1(\tilde{x}_1)\cdots\mathcal{O}_n(\tilde{x}_n)\rangle_{\mathbb{R}^d} = \Omega(x_1)^{-\Delta_1}\cdots\Omega(x_n)^{-\Delta_n}\,\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{\mathbb{R}^d}\,. \tag{2.3}$$

2.1 Conformal transformations

Let's see what is the analytic for of the conformal transformations $x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$, which imply

$$g_{\mu\nu}(\tilde{x}) = \Omega(x)^2 g_{\mu\nu}(x)$$

[5]Recall that $Z[g,J] \equiv e^{W[g,J]} = \int \mathcal{D}\Phi\; e^{-S[g,J]+\int d^dx\; J(x)\mathcal{O}(x)}$, with $\mathcal{O}$ being an operator made out of Φ fields and the convention here is Euclidean time.

Now, recalling that, in general,

$$
\begin{aligned}
g_{\mu\nu}(\tilde{x}) &= \frac{\partial x_\mu}{\partial \tilde{x}_\alpha}\frac{\partial x_\nu}{\partial \tilde{x}_\beta} g_{\alpha\beta}(x) \\
&= \left(\delta^\alpha_\mu - \partial^\alpha \epsilon_\mu\right)\left(\delta^\beta_\nu - \partial^\beta \epsilon_{\ nu}\right) g_{\alpha\beta}(x) \\
&= g_{\mu\nu}(x) - \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right).
\end{aligned}
$$

In order for this transformation to be conformal, we require

$$
\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}(x), \qquad f(x) = \frac{2}{d}\partial^\rho \epsilon_\rho. \tag{2.4}
$$

If we consider $g_{\mu\nu}(x) = \eta_{\mu\nu}$ (Minkowski metric), then differentiating the above expression, we get

$$
2\partial^2 \epsilon_\mu(x) = (2-d)\partial_\mu f(x), \tag{2.5}
$$

Finally, from (2.4) and (2.5), we arrive

$$
(d-1)\partial^2 f(x) = 0. \tag{2.6}
$$

From (2.6), we can learn two things: i) For $d = 1$, there is no constraint in the form of $f(x)$, then any smooth function is conformal; ii) For[6] $d \geq 3$ by solving $\partial^2 f(x) = 0$, we obtain that $f(x) = A + B_\mu x^\mu$, so

$$
\epsilon_\mu(x) = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho, \qquad c_{\mu\nu\rho} = c_{\mu\rho\nu} \tag{2.7}
$$

replacing the previous expression in (2.4), (2.5), we can fix $b_{\mu\nu}$ and $c_{\mu\nu\rho}$. Therefore, the conformal transformations read:

$$
\begin{aligned}
\tilde{x}^\mu &= x^\mu + a^\mu, && \text{Translations} \\
\tilde{x}^\mu &= \lambda x^\mu, && \text{Dilation} \\
\tilde{x}^\mu &= M^\mu_\nu x^\nu, && \text{Rotations} \\
\tilde{x}^\mu &= \frac{x^\mu - b^\nu x^2}{1 - 2b\cdot x + b^2 x^2}, && \text{Special Conformal}
\end{aligned} \tag{2.8}
$$

From each transformations we associate a generator and form the Lie algebra:

[6] We will study the $d = 2$ case in a separate chapter.

$$
\begin{aligned}
&[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\sigma}M^{\nu\rho}\right) \\
&[M^{\mu\nu}, P^{\sigma}] = i\left(\eta^{\mu\sigma}P^{\nu} - \eta^{\nu\sigma}P^{\mu}\right) \\
&[M^{\mu\nu}, K^{\sigma}] = i\left(\eta^{\mu\sigma}K^{\nu} - \eta^{\nu\sigma}K^{\mu}\right) \\
&[D, P^{\mu}] = -iP^{\mu} \\
&[D, K^{\mu}] = iK^{\mu} \\
&[P^{\mu}, K^{\nu}] = -2i\left(\eta^{\mu\nu}D + M^{\mu\nu}\right),
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
&P^{\mu} = -i\partial^{\mu} \\
&M^{\mu\nu} = -i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right) \\
&K^{\mu} = -i\left(x^2\partial^{\mu} - 2x^{\mu}x^{\nu}\partial_{\nu}\right) \\
&D = ix^{\mu}\partial_{\mu}.
\end{aligned}
\tag{2.10}
$$

Additionally, we have a discrete $\mathbb{Z}_2$ transformation called *Inversion* (I):

$$
I:\ x^{\mu} \to \frac{x^{\mu}}{x^2}. \tag{2.11}
$$

Why don't we consider it as part of the algebra? Simply because it maps points close to the identity to points far away from it (e.g. the origin gets mapped to infinity).

2.2 Energy-momentum tensor criterion

Recall that under an arbitrary transformation of coordinates of the form $x^{\mu} \to \tilde{x}^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$ can be considered as a translation of x^{μ}. Then the action changes by

$$
\begin{aligned}
\delta S &= \int d^d x\ T^{\mu\nu}\partial_{\mu}\epsilon_{\nu} \\
&= \frac{1}{2}\int d^d x\ T^{\mu\nu}\left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right),
\end{aligned}
\tag{2.12}
$$

where $T^{\mu\nu}$ is the energy-momentum tensor and is the current of four-vector translations. In the second line, we use the fact that the stress-energy tensor $T^{\mu\nu}$ is symmetric[7].

If our transformations are conformal, then they satisfy (2.4), so δS becomes:

$$
\begin{aligned}
\delta S &= \frac{1}{2}\int d^d x\ T^{\mu\nu}\ \frac{2}{d}\ \partial^{\rho}\epsilon_{\rho}\ g_{\mu\nu} \\
&= \frac{1}{d}\int d^d x\ T^{\mu}_{\ \mu}\partial^{\rho}\epsilon_{\rho}.
\end{aligned}
\tag{2.13}
$$

[7]Notice that for Lorentz invariant theories, if it is not symmetric, then there is always a way to find a symmetric form of $T^{\mu\nu}$. This is called the *Belinfante tensor*, and this is a consequence of the ambiguity in the definition of $T^{\mu\nu}$ from the equation $\partial_{\nu}T^{\mu\nu} = 0$.

We learn something very interesting here: **Tracelessness of the energy-momentum tensor, implies the invariance of the action under conformal transformations.** In fact, there are some theories with scaling symmetry for which $T^{\mu\nu}$ can be made traceless; if so, then conformal invariance is a consequence of dilation (and Poincaré) invariance.

2.3 Operators of a CFT

The discussion we have had so far is related to classical field theory (think about the equation (2.3) as an thermal average of a statistical system in a thermal bath or just the action on $W[g,J]$ for the zero-temperature case) and then the main operators called *primary* are defined as those that transform as (2.14). However, we would like to focus our attention to the quantum case because the most subtle and interesting parts occur at the quantum level. Here the correlation functions can be thought of by defining a ground state $|0\rangle$ or by the path integral:

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\ \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)e^{-S[\Phi]}$$

Then, a natural question to ask is how many quantum numbers are needed to specify to know the operators of the theory? We're glad you asked!

Since we can translate any point to the origin by acting with P^μ, then we will focus on transformations that leave the origin invariant. That is, dilation, rotation, and special conformal. Thus, our aim is to build a representation that has energy bounded from below. In such a way, we define the operator which is annihilated by the action of K^μ as a (quasi-)primary operator (analogous to the state $|j,j\rangle$ in $SU(2)$ annihilated by J_+). Once we have this operator, all the others (descendant operators) can be obtained by acting with P^μ (analogue to J_- in our $SU(2)$ case).

Therefore, for a primary operator, we have to specify 2 quantum numbers: the Lorentz representation R (spin) and its scaling dimension Δ. So, we label our operator as $\mathcal{O}^A_\Delta(0)$, where A labels the spin representation of the operator, and the condition of the operator to be a primary is that $[K^\mu,\mathcal{O}^A_\Delta(0)]=0$. Finally, we move it to an arbitrary spacetime point x via $\mathcal{O}^A_\Delta(x)=U(x)\mathcal{O}^A_\Delta(0)U^{-1}(x)$, here $U(x)=e^{-iP\cdot x}$.

Hence, for a conformal transformation $x^\mu\to\tilde{x}^\mu=G^\mu{}_\nu x^\nu$, a (quasi-)primary operator transforms as

$$\mathcal{O}^A_\Delta(x)\to\tilde{\mathcal{O}}^A_\Delta(x)=\left|\frac{\partial x}{\partial\tilde{x}}\right|^{\Delta/d}L^A{}_B(\mathcal{R})\mathcal{O}^B_\Delta(G^{-1}x), \tag{2.14}$$

with $\mathcal{R}$ being the orthogonal matrix $\mathcal{R}^\mu{}_\nu(x)=\Omega(x)^{-1}\frac{\partial\tilde{x}^\mu}{\partial x^\nu}$, so that $\mathcal{R}^T\eta\mathcal{R}=\eta$. And $\left|\frac{\partial\tilde{x}}{\partial x}\right|^{1/d}=\Omega(x)$ comes from the Jacobian when doing the change of coordinates.

2.4 Consequences of conformal symmetry

2.4.1 Conformal Invariants

Consider any set of points in a CFT, then

i) Translation and rotation invariance implies

$$|x_1 - x_2| = invariant.$$

ii) Including scale invariance requires

$$\frac{|x_1 - x_2|}{|x_3 - x_4|} = invariant.$$

iii) If we apply special conformal transformation to $|x_1 - x_2|$, then by (**??**)

$$|x_1 - x_2| \to \frac{|x_1 - x_2|}{(1 - 2b\cdot x_1 + b^2 x_1^2)^{1/2}(1 - 2b\cdot x_2 + b^2 x_2^2)^{1/2}}.$$

So, the conformal invariant is composed by cross ratios, for instance for 4 points we have two ratios:

$$\frac{|x_1 - x_2|\,|x_3 - x_4|}{|x_1 - x_3|\,|x_2 - x_4|}, \qquad \frac{|x_1 - x_4|\,|x_2 - x_3|}{|x_1 - x_3|\,|x_2 - x_4|}.$$

Which can be seen schematically as in Figure 3:

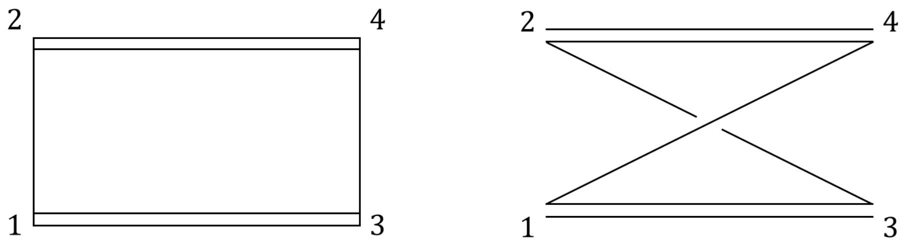


Figure 3. The two conformal ratios schematic form for 4 points.

In fact, one can show that there are $\frac{n}{2}(n-3)$ cross ratios for n distinct points (the expressions can be quite complicated).

2.4.2 Conformal correlation functions

Conformal transformations constraint the form of observables, in particular, the correlation functions of primaries. To see this, the conformal invariance implies:

$$\begin{aligned}\left\langle \mathcal{O}^{A_1}_{\Delta_1}(\tilde{x}_1)\cdots\mathcal{O}^{A_n}_{\Delta_n}(\tilde{x}_n)\right\rangle &= \left\langle \tilde{\mathcal{O}}^{A_1}_{\Delta_1}(\tilde{x}_1)\cdots\tilde{\mathcal{O}}^{A_n}_{\Delta_n}(\tilde{x}_n)\right\rangle \\ &= \Omega(x_1)^{-\Delta_1}\cdots\Omega(x_n)^{-\Delta_n} L^{A_1}{}_{B_1}(\mathcal{R})\cdots L^{A_n}{}_{B_n}(\mathcal{R})\left\langle \mathcal{O}^{B_1}_{\Delta_1}(x_1)\cdots\mathcal{O}^{B_n}_{\Delta_n}(x_n)\right\rangle.\end{aligned} \tag{2.15}$$

This is the generalization of (2.3), for any field in a representation $\mathcal{R}$ of the Lorentz group. But, for simplicity, let's focus on a theory with scalar fields only, so that (2.3) is enough.

Given that we have more symmetry than Poincaré, then that will give us constraints on the form of the $n-$point functions. Start with the 1-point function. If we apply the invariance under rotations and translations, we then have

$$\langle \mathcal{O}(x) \rangle = h\left(|x|\right),$$

now using the rescaling covariance, we require

$$h\left(|x|\right) = \lambda^{\Delta} h\left(\lambda |x|\right)$$

to fulfill this condition, $h(|x|)$ should be a polynomial in $|x|$, namely

$$\langle \mathcal{O}(x) \rangle = \frac{C}{|x|^{\Delta}}, \qquad C = \text{constant}.$$

Finally, using (**??**) and (2.3), we have

$$\frac{C}{|x|^{\Delta}} = \frac{C}{|x|^{\Delta}} \frac{(1 - 2b \cdot x + b^2 x^2)^{\Delta/2}}{(1 - 2b \cdot x + b^2 x^2)^{\Delta}},$$

since Δ can have any value (scaling dimension of the field), this expression is satisfied when $\Delta = 0$ or $C = 0$, hence

$$\langle \mathcal{O}(x) \rangle = \begin{cases} C, \ \Delta = 0, \\ 0, \ \ \Delta \neq 0 \end{cases}, \tag{2.16}$$

where the operator with scaling dimension $\Delta = 0$ is the identity $\mathbb{I}$ and $C = \langle \mathcal{O}(0) \rangle$ by translation invariance.

Let us see what happens to the two-point function. If we apply the invariance under rotations and translations, we then have

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = h\left(|x_1 - x_2|\right),$$

now using the rescaling covariance, we require

$$h\left(|x_{12}|\right) = \lambda^{\Delta_1 + \Delta_2} h\left(\lambda |x_{12}|\right)$$

with the notation $x_{12} = x_1 - x_2$. To fulfill this condition, $h(|x|)$ should be a polynomial in $|x|$, namely

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \frac{C_{12}}{|x_{12}|^{\Delta_1 + \Delta_2}}.$$

Finally, let's use (**??**) and (2.3):

$$\frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}} \frac{(1 - 2b \cdot x_1 + b^2 x_1^2)^{(\Delta_1+\Delta_2)/2} (1 - 2b \cdot x_2 + b^2 x_2^2)^{(\Delta_1+\Delta_2)/2}}{(1 - 2b \cdot x_1 + b^2 x_1^2)^{\Delta_1} (1 - 2b \cdot x_2 + b^2 x_2^2)^{\Delta_2}},$$

from this last expression, for $C_{12} \neq 0$, we have $\Delta_1 + \Delta_2 = 2\Delta_1$ and $\Delta_1 + \Delta_2 = 2\Delta_2$, that is $\Delta_1 = \Delta_2$. Notice that we can rescale the operators $\mathcal{O}$ in such a way that the coefficient

$C_{12} = 1$. In conclusion, the 2-point function in a CFT for a quasi-primary (scalar) operator is

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{1}{|x_{12}|^{2\Delta_1}}\,\delta_{\Delta_1,\Delta_2}. \tag{2.17}$$

Finally, in a similar fashion for the 3-point function we have that covariance under rotations, translations and scaling gives
$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \sum_{\substack{a,b,c\\ a+b+c=\Delta_1+\Delta_2+\Delta_3}} \frac{C_{123}^{abc}}{|x_{12}|^a|x_{23}|^b|x_{13}|^c}$, using (??) and (2.3), we have
$\frac{C_{123}^{abc}}{|x_{12}|^a|x_{23}|^b|x_{13}|^c} = \frac{C_{123}^{abc}}{|x_{12}|^a|x_{23}|^b|x_{23}|^c}\frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}}$, where $\gamma_i = 1-2b\cdot x_i + b^2 x_i^2$. Then, for non-zero C_{123}^{abc}, we require
$a+c = 2\Delta_1, a+b = 2\Delta_2, b+c = 2\Delta_3, a+b+c = \Delta_1+\Delta_2+\Delta_3$, whose unique solution is $a = \Delta_1+\Delta_2-\Delta_3$, $b = \Delta_2+\Delta_3-\Delta_1$, $c = \Delta_1+\Delta_3-\Delta_2$. Hence,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}\,|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}\,|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}. \tag{2.18}$$

Note that C_{123} is unique and cannot be fixed to be 1 since the fields were already rescaled to fix C_{12} to unity, it is called *structure constants* and are model dependent.

The rest of the correlation functions can be built out of the 1, 2, and 3-point functions. Therefore, **knowing the scaling dimensions of the (quasi-)primary operators and its structure constants, we know everything about the system.** This is a very powerful statement, all we need to do for a conformally invariant theory is to calculate the CFT-data, i.e. $\{\Delta_i, C_{123}\}$, and the rest is already fixed.

For instance, for the 4-point function (and all the other $n-$point functions, for $n \geq 4$), we need to consider the conformal ratios discussed previously[8]. Even though conformal symmetry does not fix the form of the function, it puts constraints in it. So, the 4-point function can be parametrized as follows:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \mathcal{F}\left(\left|\frac{x_{12}x_{34}}{x_{13}x_{24}}\right|, \left|\frac{x_{14}x_{23}}{x_{13}x_{24}}\right|\right)\prod_{i<j}^{4}|x_{ij}|^{\Delta/3-\Delta_i-\Delta_j}, \tag{2.19}$$

where $\Delta = \sum_i^4 \Delta_i$ and the function $\mathcal{F}(u,v)$ is not simply fixed by conformal invariance, but can be built out of 3-point functions and its descendants.

[8]For the $n < 4$ case, we cannot form cross ratios, so the procedure is the one followed before.

3 Aside: Noether Energy-Momentum tensor vs metric energy momentum tensor

Following some of the discussion in the last lecture, and for general interest, we look at two alternative but physically equivalent ways of defining a energy momentum (or stress-energy) tensor. At this point I am sure everyone is familiar with them, but there is still ongoing research surrounding this topic. This section is very close to that of the yellow book [1] so you might as well consider a reference after most equations. I have spelled out the calculations a bit more so you don't have to do them (but you should anyway).

3.1 Stress à la Noether

First, we define the canonical stress-energy tensor as resulting from Noether's theorem. We begin by considering an action

$$S = \int d^d x L(\Phi, \partial_\mu \Phi) \tag{3.1}$$

and making a transformation

$$x \to x' \tag{3.2}$$

$$\Phi(x) \to \Phi'(x') = \mathcal{F}(\Phi(x)) \tag{3.3}$$

we can consider an infinitesimal transformation such that

$$x^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \tag{3.4}$$

$$\Phi'\left(x'\right) = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \tag{3.5}$$

and we say these transformations are generated by

$$\delta_\omega \Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i\omega_a G_a \Phi(x) \tag{3.6}$$

so that

$$iG_a \Phi = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi - \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{3.7}$$

we can expand the action as

$$S' = \int d^d x \left(1 + \partial_\mu \left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)\right) \tag{3.8}$$

$$\times \mathcal{L}\left(\Phi + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}, \quad \left[\delta^\nu_\mu - \partial_\mu \left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)\right] \left(\partial_\nu \Phi + \partial_\nu \left[\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}\right]\right)\right) \tag{3.9}$$

and we realise that if we consider a global symmetry, the derivatives of ω_a are zero, and so the terms that do not contain derivatives in ω ,must be zero. Thus, we are left with

$$\delta S = -\int dx j^\mu_a \partial_\mu \omega_a \tag{3.10}$$

where

$$j_a^\mu = \left\{ \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \delta_\nu^\mu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Phi\right)} \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{3.11}$$

at this point, we make the definition

$$T_c^{\mu\nu} = \left\{ \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \delta_\nu^\mu \mathcal{L} \right\} \tag{3.12}$$

this is referred to as the canonical stress energy tensor. The conservation law for stress energy tensors is $\partial_\mu T^{\mu\nu} = 0$This tensor suffers from several pathologies; we would want a stress energy tensor to be gauge invariant, as its components are phusycal observables; for ezample, the 0ν components are four-momentum density; $P^\nu = \int d^{d-1} x T_c^{0\nu}$. Similarly, we would expect it to be symmetric if we want it to be coupled to gravity the same way a current is to a spin-1 field. Additionally, we would expect it to be traceless for scale invariant theories as previously discussed. We will look at an extremely convoluted example where all three fail: 4d pure electrodynamics Let us begin with the Lagrangian

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{3.13}$$

Then, the stress energy tensor is

$$T^{\mu\nu} = F^{\mu\alpha} \partial^\nu A_\alpha - \eta^{\mu\nu} \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \tag{3.14}$$

Let us check gauge invariance: it is clear that the stress energy tensor will change by $F^{\mu\alpha} \partial^\nu \partial_\alpha \Lambda$ under $A_\alpha \to \partial_\alpha \Lambda$, which is not zero. Secondly, we look at the tracelessness we would expect at the classical level from electromagnetism; $T^\mu \mu = -F^{\alpha\beta} \partial_\alpha A_\beta$ in d=4, which is also not zero. It is also apparent that this is not a symmetric tensor. There is a way to symmetrise this tensor consistently: let us consider an infinitesimal Lorentz transform under which

$$\frac{\delta x^\rho}{\delta \omega_{\mu\nu}} = \frac{1}{2} \left(\eta^{\rho\mu} x^\nu - \eta^{\rho\nu} x^\mu \right) \quad , \quad \frac{\delta \mathcal{F}}{\delta \omega_{\mu\nu}} = -i \frac{1}{2} S^{\mu\nu} \Phi \tag{3.15}$$

and so

$$j^{\mu\nu\rho} = T_c^{\mu\nu} x^\rho - T_c^{\mu\rho} x^\nu + \frac{1}{2} i \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Phi\right)} S^{\nu\rho} \Phi \tag{3.16}$$

and strive for a stress energy tensor such that

$$j^{\mu\nu\rho} = T_B^{\mu\nu} x^\rho - T_B^{\mu\rho} x^\nu \tag{3.17}$$

which makes the tensor symmetric via the conservation laws of the stress energy tensor and the current. Considering $\partial_\mu j^{\mu\nu\rho} = 0$ thus means

$$\partial_\alpha (B^{\alpha\rho\nu} - B^{\alpha\nu\rho} + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\nu\rho} \Phi) = 0 \tag{3.18}$$

it is then straightforward to check that

$$B^{\mu\rho\nu} = \frac{1}{4} i \left\{ \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Phi\right)} S^{\nu\rho} \Phi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\rho \Phi\right)} S^{\mu\nu} \Phi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \Phi\right)} S^{\mu\rho} \Phi \right\} \tag{3.19}$$

solves this constraint, and thus symmetrises the stress energy tensor. This is called the Belinfante tensor, and allows us to write

$$T_B^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} \quad , \quad B^{\rho\mu\nu} = -B^{\mu\rho\nu} \tag{3.20}$$

where the antisymmtry ensures the good health of the conservation law.

3.2 Stress from metrics

Considering a coordinate transformation $x'^\mu \to x^\mu + \epsilon^\mu(x)$, the induced change in the action is

$$\begin{aligned} \delta S &= \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu \\ &= \frac{1}{2} \int d^d x T^{\mu\nu} \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \right) \end{aligned} \tag{3.21}$$

we also have a change in the metric

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\ &= \left(\delta_\mu^\alpha - \partial_\mu \epsilon^\alpha \right) \left(\delta_\nu^\beta - \partial_\nu \epsilon^\beta \right) g_{\alpha\beta} \\ &= g_{\mu\nu} - \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \right) \end{aligned} \tag{3.22}$$

and so we can have the more usual definition

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \tag{3.23}$$

we can connect the canonical and the metric stress energy tensor through a Belinfante tensor, and their difference is [16]

$$\int d^3 x \left(T^{0\nu} - \mathcal{T}^{0\nu} \right) = \int d^3 x \partial_\alpha B^{\alpha 0 \nu} = \int d^3 x \partial_k B^{k 0 \nu} = \int_W dS_k B^{k 0 \nu} \tag{3.24}$$

over a remote surface, this difference vanishes for well behaved fields.

4 Ward Identities

We know from theorems such as Ehrnfest theorem that many of the conservation laws in classical physics are only obeyed at an expectation value level quantum mechanically. Ward identities examplify this for conserved currents inside correlators of fields. We will review the derivation of Ward identities and calculate the Ward identities in the conformal group. This chapter also follows closely from the relevant sections of the yellow book. We begin by considering an infinitesimal transformation of a field :

$$\Phi'(\boldsymbol{x}) = \Phi(x) - i\omega_a G_a \Phi(x) \tag{4.1}$$

where ω_a are constant parameters. We take an operator X which contains this field at different points in space ($X = \Phi(x_1)...\Phi(x_n)$), and its variation under the transformation as δX, and assuming that the integration measure does not change, we have

$$\langle X\rangle = \frac{1}{Z}\int \left[d\Phi'\right](X+\delta X)\exp\left\{-S[\Phi] - \int dx \partial_\mu j_a^\mu \omega_a(x)\right\} \tag{4.2}$$

and to first order in ω_a, since the X's do not depend on x, we can move it into the derivative:

$$\langle \delta X\rangle = \int dx \partial_\mu \left\langle j_a^\mu(x) X\right\rangle \omega_a(x) \tag{4.3}$$

similarly, using the definition of G_a, we can expand δX as

$$\begin{aligned}\delta X &= -i\sum_{i=1}^{n}\left(\Phi\left(x_1\right)\cdots G_a\Phi\left(x_i\right)\cdots\Phi\left(x_n\right)\right)\omega_a\left(x_i\right) \\ &= -i\int dx\omega_a(x)\sum_{i=1}^{n}\left\{\Phi\left(x_1\right)\cdots G_a\Phi\left(x_i\right)\cdots\Phi\left(x_n\right)\right\}\delta\left(x-x_i\right)\end{aligned} \tag{4.4}$$

resulting in the local version of Ward's identity:

$$\begin{aligned}&\frac{\partial}{\partial x^\mu}\left\langle j_a^\mu(x)\Phi\left(x_1\right)\cdots\Phi\left(x_n\right)\right\rangle \\ &\quad = -i\sum_{i=1}^{n}\delta\left(x-x_i\right)\left\langle\Phi\left(x_1\right)\cdots G_a\Phi\left(x_i\right)\cdots\Phi\left(x_n\right)\right\rangle\end{aligned} \tag{4.5}$$

this gives rise to conserved charges

$$Q_a = \int d^{d-1}x j_a^0(x) \tag{4.6}$$

which we will now show that generate translations. Start by taking the first field, and assuming it has the lowest time coordinate $t = x_1^0$, then surrounding this time sheet by an infinitely space-like pillbox at time $t_- < t$ and t_+. We can integrate the left hand side of the Ward identity over this surface. The sides of the pillbox will be zero if the fields drop off quickly enough, and so we will simply have the integral evaluated at the two spacelike slices. Looking at the definition of conserved charges, we see this results in

$$\left\langle Q_a\left(t_+\right)\Phi\left(x_1\right)Y\right\rangle - \left\langle Q_a\left(t_-\right)\Phi\left(x_1\right)Y\right\rangle = -i\left\langle G_a\Phi\left(x_1\right)Y\right\rangle \tag{4.7}$$

where Y is X with the first field removed. Since expectation values in QFT come with time ordering, we can flip the order of the fields of the second term at no cost, thus resulting in

$$\langle 0|\left[Q_a, \Phi\left(x_1\right)\right] Y|0\rangle = -i\langle 0|G_a \Phi\left(x_1\right) Y|0\rangle \tag{4.8}$$

and since this is true for arbitrary Y, we conclude that

$$\left[Q_a, \Phi\right] = -iG_a\Phi \tag{4.9}$$

so the conserved charge actually generates the symmetry transforms. Those of you who took QFT II with Ira will have seen this result before. A last note is that the derivation is done in Euclideanised space, to go back to Mionkoski, it suffices to send $Q \to -iQ$

4.1 Ward Identities in a CFT

We have several generators of symmetry in CFTs, and we will see how each one gives rise to different conservation laws, and thus different Ward identities. Firstly, we consider a translation $x^\mu \to x^\mu + \epsilon^\mu$. According to 3.11 $j^{\mu\nu} = T^{\mu\nu}$, and thus the ward identity becomes

$$\partial_\mu \left\langle T^\mu_v X\right\rangle = -\sum_i \delta\left(x - x_i\right) \frac{\partial}{\partial x_i^\nu}\langle X\rangle \tag{4.10}$$

given the generator $P_\nu = -i\partial_\nu$ For rotations, we have the transformations for fields and coordinates as follows:

$$x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu \tag{4.11}$$

and

$$\mathcal{F}(\Phi) = L_\Lambda \Phi \simeq 1 + \frac{1}{2} i\omega_{\rho\nu} S^{\rho\nu} \tag{4.12}$$

from which we can deduce that $\frac{\delta x^\mu}{\delta\omega_{\rho\nu}} = \frac{1}{2}\left(\eta^{\rho\mu}x^\nu - \eta^{\nu\mu}x^\rho\right)$. Using the previous definition of the generator

$$iG_a\Phi = \frac{\delta x^\mu}{\delta\omega_a}\partial_\mu\Phi - \frac{\delta\mathcal{F}}{\delta\omega_a} \tag{4.13}$$

we can see that the generator or Lorentz transformations is

$$L^{\rho\nu} = i\left(x^\rho\partial^\nu - x^\nu\partial^\rho\right) + S^{\rho\nu} \tag{4.14}$$

and the conserved current is

$$j^{\mu\nu\rho} = T_c^{\mu\nu}x^\rho - T_c^{\mu\rho}x^\nu + \frac{1}{2} i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)} S^{\nu\rho}\Phi \tag{4.15}$$

although for the Ward identities we shall use the Belinfante version of the Conserved current;

$$j^{\mu\nu\rho} = T_B^{\mu\nu}x^\rho - T_B^{\mu\rho}x^\nu \tag{4.16}$$

so the Ward identity becomes

$$\partial_\mu \left\langle\left(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu\right) X\right\rangle = \sum \delta\left(x - x_i\right)\left[\left(x_i^\nu\partial_i^\rho - x_i^\rho\partial_i^\nu\right)\langle X\rangle - iS_i^{\nu\rho}\langle X\rangle\right] \tag{4.17}$$

we can exapnd the l.h.s. as

$$\langle(\partial_\mu T^{\mu\nu}x^\rho + T^{\rho\nu} - x^\nu \partial_\mu T^{\mu\rho} - T^{\nu\rho})X\rangle \tag{4.18}$$

$$= -\sum_i \delta(x-x_i)\left[x^\rho \frac{\partial}{\partial x_i^\nu}\langle X\rangle - x^\nu \frac{\partial}{\partial x_i^\rho}\langle X\rangle\right] + \langle(T^{\rho\nu} - T^{\nu\rho})X\rangle \tag{4.19}$$

where for the second line we used the first Ward identity we had derived. Now we see that most of the terms cancel, leaving us with

$$\langle\left(T^{\rho\nu} - T^{\nu\rho}\right)X\rangle = -i\sum_i \delta\left(x-x_i\right) S_i^{\nu\rho}\langle X\rangle \tag{4.20}$$

telling us that for Lorentz invariance, the constraint is that the expectation value of Stress-energy tensor is symmetric apart from points of where fields overlap in the correlator. Finally, dilations have $x^\mu \to x^\mu + \lambda x^\mu$ and so $J_D^\mu = T^{\mu\nu}x_\nu$ and since $\mathcal{F}(\Phi) = (1-\lambda\Delta)\Phi$, the generator of dilations is

$$G_D = (-ix^\nu\partial_\nu - i\Delta) \tag{4.21}$$

and so the ward identity becomes

$$\partial_\mu \left\langle T^\mu{}_\nu x^\nu X\right\rangle = -\sum_i \delta\left(x-x_i\right)\left\{x_i^\nu \frac{\partial}{\partial x_i^\nu}\langle X\rangle + \Delta_i\langle X\rangle\right\} \tag{4.22}$$

acting on the derivative, we find

$$\langle T^\mu{}_\mu X\rangle = -\sum_i \delta\left(x-x_i\right)\Delta_i\langle X\rangle \tag{4.23}$$

which shows the tracelessness of the stress energy tensor when sandwiched in propagators.

4.2 A "Weird" identity

Erick and I were discussing that it appears to be thee case that most of the texts online only derive Ward identities for the translations, Lorentz, and scale transformations of the conformal group, but not the special conformal transformations. To this end I will derive this Ward identity, and dazzlingly enough, show that it is trivial when using the other three. We will work with spin) fields to make our lives easier We begin by looking at the transformation law for fields under special conformal transformations. Under a transformation 2.8, the scale factor is $\Lambda(x) = (1-2(b\cdot x)+(b\cdot b)(x\cdot x))^2$. You should derive this result, though as I know you won't, I'll leave a derivation in the appendix. I would encourage you to struggle with it for at least half an hour before looking at the solution. With this in mind, to first order since

$$\phi \to \phi'(x') = \frac{\partial x'}{\partial x}^{-\frac{Delta}{d}}\phi(x) \tag{4.24}$$

$$= (\Lambda^{-\frac{d}{2}})^{-\frac{\Delta}{d}}\phi(x) \tag{4.25}$$

$$= \Lambda^{\Delta/2}\phi(x) \tag{4.26}$$

we have

$$\frac{\partial \mathcal{F}}{\partial b^\nu} = -2\Delta x_\nu \phi(x) \tag{4.27}$$

meanwhile, to first order in b we also have

$$\frac{\partial x^\mu}{\partial b^\nu} = -\delta^\mu_\nu x^2 + 2x_\nu x^\mu \tag{4.28}$$

and so the generator is

$$iG_\nu \phi = (-\delta^\mu_\nu x^2 + 2x_\nu x^\mu)\partial_\mu \phi - 2\Delta x_\nu \phi(x) \tag{4.29}$$

This means that the conserved current is

$$\mathcal{J}^\mu{}_\nu = T^{\mu\alpha}(-g_{\alpha\nu}x^2 + 2x_\alpha x_] \nu) \tag{4.30}$$

and so the Ward identity becomes

$$\frac{\partial}{\partial x^\mu} \left\langle \left(-T^\mu_\nu x^2 + 2T^{\mu\alpha}x_\alpha x_\nu\right) X \right\rangle \tag{4.31}$$

$$= -i \sum_i \delta(x - x_i) \left\langle \phi_1 \cdots \left(\left(-\delta^\mu_\nu x^2 + 2x_\nu x^\mu\right) \partial_\mu - 2\Delta x_\nu \phi(x)\right) \phi_i(x_i) \cdots \phi(x_n)\right\rangle \tag{4.32}$$

We can take the derivatives of the left hand side to get

$$-x^2\partial_\mu \langle T^\mu{}_\nu X\rangle - 2\langle T^\mu{}_\nu x_\mu X\rangle + 2x_\alpha x_\nu \partial_\mu \langle T^{\mu\alpha} X\rangle + 2\langle T^\mu{}_\mu x_\nu X\rangle + 2\langle T^{\mu\alpha} x_\alpha X\rangle \tag{4.33}$$

using the previous three Ward identities, and since we are working wit spin 0 fields, we see that the one corresponding to lorentz transformations is zero as $S^{\mu\nu} = 0$ for spin 0 fields, we can write this as

$$-i \sum_i \delta(x - x_i)(-x^2\partial_{\nu i} + 2x_\alpha x_\nu \partial_{\alpha i} - 2x_\nu \Delta_i)\langle X\rangle \tag{4.34}$$

which is the same as what we have in the right hand side. Thus, this Ward identity does not bring us anything new.

5 CFT in 2d

Some of the most studied theories are CFTs in 2D because of amyriad of reasons. Some of these include, as we shall soon see, the fact that the algebra in 2D is infinite dimensional, often referred to as the Virasoro algebra, making it a much richer theory than in other dimensions. This also gives rise to the conformal bootstrap, a way of determining the properties of the theory based on the constraints coming from said algebra. There are many other good reasons why 2d CFTs are studied, from string theory, to statistical mechanics systems. The following section is once again a condensed version of the yellow book.

5.1 2D conformal maps

The conformal transformations in 2D take a particularly simple form in complex coordinates. To see this, we consider coordinates $z^\mu = (z^0, z^1)$ and a transformations $z^\mu \to w^\mu$

$$g^{\mu\nu} \to \left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\nu}{\partial z^\beta}\right) g^{\alpha\beta}$$

We again use that the new metric should be proportional to the original one after a conformal transformation; $g'_{\mu\nu}(w) \propto g_{\mu\nu}(z)$, leading to

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 \tag{5.2}$$

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0 \tag{5.3}$$

These conditions are equivalent either to

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \tag{5.4}$$

or to

$$\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \tag{5.5}$$

We see that this is simply telling us that the functions must be holomorphic or anti-holomorphic. A good way to express this is using the so-called light-cone coordinate:

$$\begin{aligned} z &= z^0 + iz^1 & z^0 &= \tfrac{1}{2}(z+\bar{z}) \\ \bar{z} &= z^0 - iz^1 & z^1 &= \tfrac{1}{2i}(z-\bar{z}) \\ \partial_z &= \tfrac{1}{2}(\partial_0 - i\partial_1) & \partial_0 &= \partial_z + \partial_{\bar{z}} \\ \partial_{\bar{z}} &= \tfrac{1}{2}(\partial_0 + i\partial_1) & \partial_1 &= i(\partial_z - \partial_{\bar{z}}) \end{aligned}$$

We shall sometimes write $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$ when there is no ambiguity about the differentiation variable. In terms of the coordinates z and $\bar{z}$, the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{5.7}$$

where the index μ takes the values z and $\bar{z}$, in that order. This metric tensor allows us to transform a covariant holomorphic index into a contravariant antiholomorphic index and vice versa. The antisymmetric tensor $\varepsilon_{\mu\nu}$ in holomorphic form is

$$\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix} \quad \varepsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}$$

In this language, the holomorphic Cauchy-Riemann equations become simply

$$\partial_{\bar{z}} w(z,\bar{z}) = 0 \tag{5.9}$$

whose solution is any holomorphic mapping (no $\bar{z}$ dependence):

$$z \to w(z) \tag{5.10}$$

This splits the complexified tangent bundle into holomorphic and antiholomorphic parts:

$$TM \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M \tag{5.1}$$

this means that the manifold must have a complex structure. In 2D, this roughly translates to orinetability, but its implications are far reaching. For starters, it means we can decompose fields into $\Phi(z,\bar{z}) = \Phi_L(z) \otimes \Phi_R(z)$, exhibiting independent dynamics. As we will see, this leads to two copies of the algebra, and the weights of the algebra will also decompose into $\Delta = h + \bar{h}$. The far reaching consequences will become evident as we move through the section.

It is a known fact that complex endomorphisms of $\mathbb{C}$ are conformal maps,

$$dw = \left(\frac{dw}{dz}\right) dz \tag{5.11}$$

meaning that the conformal group in 2D is the set of these maps. Each coefficient in the Laurent series determines a generator of conformal symmetries, and thus makes the conformal group infinite dimensional.

5.2 Local vs global conformal group

The endomorphisms that form the conformal group must be invertible by definition of a group. Nonetheless, it is often the case that topology can present obstructions to global trivialisations of these fields. More usual examples of this that come to mind will be Dirac monopoles arising from non-trivial U(1) bundles over S^2 as we cannot define the gauge field in both hemispheres (globally). This is a similar behaviour to the one we see in the conformal maps; we must divide them into the ones that can be defined globally, and the ones that can only be defined locally. The former is referred to as the Special conformal group, and can be defined as

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1 \tag{5.2}$$

where a,b,c,d are complext numbers, and the so called projective transformations.

We can repackage the numbers into a matrix with unit determinant

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and thus the global conformal group in 2D is isomorphic to $SL(2, \mathbb{C})$, which is isomorphic to the Lorentz, group. To see why this is the only possible transformation, we should not have any branch point or essential singularity. This is because at branch points the function is multi-valued, something that would kill the invertibility condition. Similarly, essential singularities have wild behaviour near the poles such as e^1/z making them not well defined, and thus disrupting the group behaviour. The hallmark of the latter is Laurent series with infinite negative powers.Thus, only poles are admissible, and thus, the function must be a ratio of polynomials $f(z) = \frac{P(z)}{Q(z)}$, but if P(z) is of order higher than one, the neighbourhood of the solution gets wrapped around several times around the origin, and thus cannot be inverted. Thus, since the same argument goes for Q(z), they both must be linear, yielding the above result.

5.3 Conformal generators

We now look at local generators that need not be defined globally. These are the transformations of $z' = z + \epsilon(z) \quad \epsilon(z) =_{-\infty}^{\infty} c_s z^{n+1}$ where we are Laurent expanding around the origin. This leads to the expansion

$$\begin{aligned} \phi'\left(z', \bar{z}'\right) &= \phi(z, \bar{z}) \\ &= \phi\left(z', \bar{z}'\right) - \epsilon\left(z'\right) a' \phi\left(z', \bar{z}'\right) - \bar{\epsilon}\left(\bar{z}'\right) \bar{a}' \phi\left(z', \bar{z}'\right) \end{aligned} \tag{5.16}$$

or

$$\begin{aligned} \delta\phi &= -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi \\ &= \sum_n \left\{ c_n \ell_n \phi(z, \bar{z}) + \bar{c}_n \bar{\ell}_n \phi(z, \bar{z}) \right\} \end{aligned} \tag{5.17}$$

where we have introduced the generators

$$\ell_n = -z^{n+1}\partial_z \quad \bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{5.18}$$

it is a simple calculation to show that the generators obey the following algebra

$$\begin{aligned} &[\ell_n, \ell_m] = (n-m)\ell_{n+m} \\ &\left[\bar{\ell}_n, \bar{\ell}_m\right] = (n-m)\bar{\ell}_{n+m} \\ &\left[\ell_n, \bar{\ell}_m\right] = 0 \end{aligned} \tag{5.19}$$

which, as promised previously, factorises into two identical algebras that form a product group. This algebra is also called the Witt algebra. The subalgebras formed by l, l_{-1}, l_1

form the global conformal group; $l_{-1} = -\partial_z$ represents translations, $l_0 = -z\partial_z$ represents rotations and scale transformations, and l_1 generates special conformal transforms. A caveat to mention is that z and $\bar{z}$ are in principle independent variables, though in 2d we only have two coordinates, and so it would seem like we have an extra two. To fix this, we must take the hypersurface in $z, \bar{z}$ space where $\bar{z} = z^*$. In particular, the generators that preserve the real surface $z_0, z_1 \in \mathbb{R}$ are the linear combinations

$$\ell_n + \bar{\ell}_n \text{ and } i\left(\ell_n - \bar{\ell}_n\right) \tag{5.20}$$

In particular, $\ell_0 + \bar{\ell}_0$ generates dilations on the real surface, and $i\left(\ell_0 - \bar{\ell}_0\right)$ generates rotations.

5.1.4. Primary Fields

Given a field with spin s and scaling dimension Δ, we define the holomorphic conformal dimension as

$$h = \frac{1}{2}(\Delta + s) \quad \bar{h} = \frac{1}{2}(\Delta - s) \tag{5.21}$$

and a quasi-primary field transforms as

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \tag{5.22}$$

The above shows that a quasiprimary field of conformal dimensions ($h, \bar{h}$) transforms like the component of a covariant tensor of rank $h + \bar{h}$ having h " z " indices and $\bar{h}$ " z " indices.

We can expand the map around the identity; $w = z + \epsilon(z)$

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\phi &\equiv \phi'(z, \bar{z}) - \phi(z, \bar{z}) \\ &= -\left(h\phi\partial_z\epsilon + \epsilon\partial_z\phi\right) - \left(\bar{h}\phi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\phi\right) \end{aligned} \tag{5.23}$$

To answer Margit from last week too, the difference between primary and quasi-primary fields is that primary fields transform like above under conformal transformations, while quasi-primary transformations only have these properties under global conformal transformations. Fields that are not primary are called secondary. The transformation rule for fields becomes

$$\begin{aligned} &\left\langle \phi_1\left(w_1, \bar{w}_1\right) \ldots \phi_n\left(w_n, \bar{w}_n\right)\right\rangle = \\ &\prod_{i=1}^{n} \left(\frac{dw}{dz}\right)_{w=w_i}^{-h_i} \left(\frac{d\bar{w}}{d\bar{z}}\right)_{\bar{w}=\bar{w}_i}^{-\bar{h}_i} \left\langle \phi_1\left(z_1, \bar{z}_1\right) \ldots \phi_n\left(z_n, \bar{z}_n\right)\right\rangle \end{aligned} \tag{5.24}$$

5.4 Ward identities, again

Recall we derived the conformal Ward identities last lesson

$$
\begin{aligned}
\frac{\partial}{\partial x^{\mu}}\left\langle T^{\mu}{}_{\nu}(x) X\right\rangle &=-\sum_{i=1}^{n} \delta\left(x-x_{i}\right) \frac{\partial}{\partial x_{i}^{\nu}}\langle X\rangle \\
\varepsilon_{\mu\nu}\left\langle T^{\mu\nu}(x) X\right\rangle &=-i \sum_{i=1}^{n} s_{i} \delta\left(x-x_{i}\right)\langle X\rangle \\
\left\langle T_{\mu}^{\mu}(x) X\right\rangle &=-\sum_{i=1}^{n} \delta\left(x-x_{i}\right) \Delta_{i}\langle X\rangle .
\end{aligned} \tag{5.32}
$$

Now, before diving into computations, we will derive a useful formula that is valid distributionally (when integrated over a two-chain). The claim is that $\partial_{\bar{z}} \frac{1}{z}=2\pi\delta(x)\delta(y)$

So lets show that this is the case. I use the proof from [17] but I flesh out the details that I had to work through for it to be clear.

Claim(Cauchy–Pompeiu): Let $U \subset \mathbb{C}$ be a bounded open set with piecewise-C^1 boundary ∂U oriented positively (see appendix B), and let $f: \overline{U} \to \mathbb{C}$ be continuous with bounded continuous partial derivatives in U. Then for $z \in U$:

$$
f(z)=\frac{1}{2\pi i}\int_{\partial U}\frac{f(\xi)}{\xi-z}d\xi+\frac{1}{2\pi i}\int_{U}\frac{\partial f}{\partial\bar{\xi}}(\xi)\frac{d\xi\wedge d\bar{\xi}}{\xi-z}. \tag{4.1.1}
$$

Proof:

Fix $z \in U$. We wish to apply Stokes' theorem, but the integrand is not smooth at z. Let $\Delta_r(z)$ be a small disc such that $\Delta_r(z) \subset\subset U$. Stokes' theorem now applies on $U \setminus \Delta_r(z)$.

We can now use the generalised stokes theorem:

$$
\int_{\partial\mathcal{M}}\omega=\int_{\mathcal{M}}d\omega \tag{5.3}
$$

applying this to the following equation

$$
\int_{\partial U}\frac{f(\xi)}{\xi-z}d\xi-\int_{\partial\Delta_r(z)}\frac{f(\xi)}{\xi-z}d\xi=\int_{U\setminus\Delta_r(z)}d\left(\frac{f(\xi)}{\xi-z}d\xi\right)=\int_{U\setminus\Delta_r(z)}\frac{\partial f}{\partial\bar{\xi}}(\xi)\frac{d\xi\wedge d\bar{\xi}}{\xi-z}. \tag{4.1.2}
$$

The second equality follows because holomorphic derivatives in ξ have a $d\bar{\xi}$, and when we wedge them with $d\xi$, we just get zero. We now wish to let the radius r go to zero. Via the exercise above, $\frac{\partial f}{\partial\bar{\xi}}(\xi)\frac{d\xi\wedge d\bar{\xi}}{\xi-z}$ is integrable over all of U. Therefore,

$$
\lim_{r\to 0}\int_{U\setminus\Delta_r(z)}\frac{\partial f}{\partial\bar{\xi}}(\xi)\frac{d\xi\wedge d\bar{\xi}}{\xi-z}=-\int_{U}\frac{\partial f}{\partial\bar{\xi}}(\xi)\frac{d\xi\wedge d\bar{\xi}}{\xi-z}. \tag{4.1.3}
$$

The second equality is simply swapping the order of $d\xi$ and $d\bar{\xi}$. By continuity of f,

$$
\lim_{r\to 0}\frac{1}{2\pi i}\int_{\partial\Delta_r(z)}\frac{f(\xi)}{\xi-z}d\xi=\lim_{r\to 0}\frac{1}{2\pi}\int_0^{2\pi}f(z+re^{i\theta})d\theta=f(z). \tag{4.1.4}
$$

where we have first taken $\xi\to\xi+z$ and then used $\partial_\theta z=iz$ to go from one equality to the other.

The theorem follows from this. Now we can apply this to an arbitrary function as follows; consider a (not neccessarily holomorphic) function f with no support in $\partial\mathcal{M}$

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial\mathcal{M}} \frac{f(z)dz}{z-\zeta} - \frac{1}{\pi}\iint_{\mathcal{M}} \frac{\partial f(z)}{\partial \bar{z}}\frac{dx\wedge dy}{z-\zeta}, \tag{*}$$

then the integral over the bondary is zero due to the lack of support, and, while the area integral $\iint_D$ states that

$$\frac{\partial}{\partial \bar{z}}\frac{1}{z-\zeta} = \pi\delta^{(2)}(z-\zeta).$$

as a density. This is equivalent to the result promised $\Box$. The same result is obtained exchanging $\bar{z} \leftrightarrow z$. There is a similar proof in [1] but this one is more general as it does not require the function to be holomorphic and it teaches you (me) the Cauchy–Pompeiu theorem. We can apply these to the Ward identities, which become

$$2\pi\partial_z \langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}} \langle T_{zz}X\rangle = -\sum_{i=1}^{n} \partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle \tag{5.4}$$

$$2\pi\partial_z \langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_z \langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n} \partial_z\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle \tag{5.5}$$

$$2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n} \delta\left(x-x_i\right)\Delta_i\langle X\rangle \tag{5.6}$$

$$-2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n} \delta\left(x-x_i\right)s_i\langle X\rangle \tag{5.7}$$

we can easily combine the latter two as

$$2\pi\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n} \partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle X\rangle \tag{5.8}$$

$$2\pi\langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n} \partial_z\frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i\langle X\rangle \tag{5.9}$$

we can now re-use these in the first equations to find

$$\partial_{\bar{z}}\left\{\langle T(z,\bar{z})X\rangle - \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right]\right\} = 0 \tag{5.10}$$

$$\partial_z\left\{\langle \bar{T}(z,\bar{z})X\rangle - \sum_{i=1}^{n}\left[\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2}\langle X\rangle\right]\right\} = 0 \tag{5.11}$$

where we have introduced a renormalized energy-momentum tensor

$$T = -2\pi T_{zz} \quad \bar{T} = -2\pi T_{\bar{z}\bar{z}}$$

Thus the expressions between braces in (5.10) and (5.11) are respectively holomorphic and antiholomorphic, we may write

$$\langle T(z)X\rangle = \sum_{i=1}^{n}\left\{\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right\} + \text{ reg.} \tag{5.12}$$

where "reg." stands for a holomorphic function of z, regular at $z = w_i$. There is a similar expression for the antiholomorphic counterpart. Keep in mind that this relies on the assumption that $T(z)$ is everywhere well-defined, and so $T(0)$ should be finite.

6 Operator Product Expansion

Typically, correlation functions have singularities when the positions of 2 or more points coincide. The Operator Product Expansion (OPE) is a representation of a product of operators at distinct points x and y by a sum of single well-defined operators as $x \to y$, multiplied by a function of $(x-y)$. One can interpret this as a multipole expansion, as seen from far away, that is

$$\begin{aligned}
\mathcal{O}_1(0)\mathcal{O}_2(x) &= \sum_{\substack{\text{all operators}\\ i}} C_i(x)\mathcal{O}_i(0)\\
&= \sum_{\substack{\text{primaries}\\ \kappa}}\left[\underbrace{C_\kappa(x)\mathcal{O}_\kappa(0)}_{\text{primaries}} + \underbrace{C^\mu_\kappa(x)\partial_\mu\mathcal{O}_\kappa(0) + C^{\mu\nu}_\kappa(x)\partial_\mu\partial_\nu\mathcal{O}_\kappa(0) + \ldots}_{\text{descendants}}\right],
\end{aligned}$$

where $x \ll 1$. We should be aware that the OPE is meaningful only within correlation functions.

What about C_κ, can we predict them? Let's see, consider the 3-point function:

$$\begin{aligned}
\langle\mathcal{O}_1(0)\mathcal{O}_2(x)\mathcal{O}_\kappa(y)\rangle &= \frac{C_{12\kappa}}{|x|^{\Delta_1+\Delta_2-\Delta_\kappa}|y|^{2\Delta_\kappa}}\left(1-\frac{2x\cdot y}{y^2}+\frac{x^2}{y^2}\right)^{\frac{\Delta_1-\Delta_2-\Delta_\kappa}{2}}\\
&= \frac{C_{12\kappa}}{|x|^{\Delta_1+\Delta_2-\Delta_\kappa}|y|^{2\Delta_\kappa}}\left(1-(\Delta_1-\Delta_2-\Delta_\kappa)\frac{x\cdot y}{y^2}+\cdots\right)\\
&= \sum_{\kappa'}\left[C_{\kappa'}(x)\langle\mathcal{O}_{\kappa'}(0)\mathcal{O}_\kappa(y)\rangle + C^\mu_{\kappa'}(x)\partial_\mu\langle\mathcal{O}_{\kappa'}(0)\mathcal{O}_\kappa(y)\rangle+\cdots\right]\\
&= \frac{C_\kappa(x)}{|y|^{2\Delta_\kappa}} - 2\Delta_\kappa\frac{C^\mu_\kappa(x)\ y_\mu}{|y|^{2\Delta_\kappa}|y|^2}+\cdots
\end{aligned} \tag{6.1}$$

where in the second line, we've Taylor expand for $x \ll 1$ and in the third line we've used the OPE. It is straightforward to identify:

$$C_\kappa(x) = \frac{C_{12\kappa}}{|x|^{\Delta_1+\Delta_2-\Delta_\kappa}}, \qquad C^\mu_\kappa(x) = \frac{C_{12\kappa}}{|x|^{\Delta_1+\Delta_2-\Delta_\kappa}}\frac{(\Delta_1-\Delta_2-\Delta_\kappa)\ x^\mu}{2\Delta_\kappa}, \qquad \cdots \tag{6.2}$$

Thus, for example, the four-point function can be written as:

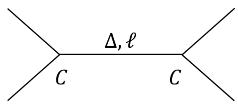
$$\begin{aligned}\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_4(x_4)\rangle &\underset{\text{(OPE)}}{=} \sum_\kappa C_{12\kappa}C_{\kappa 34}\left[\sum_{n=0}^{\infty} b^{\mu_1\cdots\mu_n}(x_2-x_1)\partial_{x_1^{\mu_1}}\cdots\partial_{x_n^{\mu_n}}\underbrace{\left(\frac{\langle\mathcal{O}_\kappa^{(\Delta_\kappa,\ell)}\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle}{C_{\kappa 34}}\right)}_{\text{fixed by conformal symmetry}}\right]\\ &= \sum_\kappa C_{12\kappa}C_{\kappa 34}\ \mathrm{F}_\kappa^{(\Delta_\kappa,\ell)}(x_1,\ldots,x_4),\end{aligned}$$

where κ represents the set of primaries with scaling dimension Δ_κ and spin ℓ, and $\mathrm{F}_\kappa^{(\Delta,\ell)}(x_1,\ldots,x_4)$ is called *conformal blocks*.

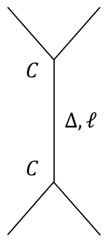
Remember when we told you that knowing the structure constants $C_{ij\kappa}$ and the scaling dimensions Δ_κ allows you to calculate any other correlation function in your theory? Yes? Well, this is what we meant. Because, we can use the OPE to write any correlation function as a sum of 3 point functions.

If you notice, for this example, we took the structure constants $C_{12\kappa}$ and $C_{\kappa 34}$, what if instead we use $C_{13\kappa}$ and $C_{\kappa 24}$? This will not change the final result. So, we have an associativity property, which pictorially looks like:

$$\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_4(x_4)\rangle = \sum$$



$$= \sum$$



(6.3)

The last two equations are called *Bootstrap equations*, and sometimes people say that a CFT is a set of $\{\Delta_i, C_{ijk}\}$, which obey the Bootstrap equations (as an alternative definition of what a CFT is).

Coming back to 2 dimensions, the OPE of the stress energy tensor with a primary field ϕ is written from (5.12) as:

$$\begin{aligned}T(z)\phi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\ \partial_w\phi(w,\bar{w})\\ \bar{T}(\bar{z})\phi(w,\bar{w}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\ \partial_{\bar{w}}\phi(w,\bar{w}),\end{aligned} \tag{6.4}$$

where here $\sim$ means equality up to regular functions as $z\to w$, and again, this expansion makes sense only within correlation functions.

7 Examples

In this section, we'll work out some examples of CFTs and discuss some of its consequences.

7.1 The free boson in $d \neq 2$ dimensions

One of the most studied theories is the free scalar field theory, this theory has a conformal symmetry as well. To warm up, let's calculate its 2-point function in the usual way and compare with our predictions. The action of the theory is

$$\begin{aligned} S &= \frac{1}{2}\int d^d x \; \partial_\mu \phi \partial^\mu \phi \\ &= \frac{1}{2}\int d^d x \; d^d y \; \phi(x) A(x,y) \phi(y), \end{aligned} \tag{7.1}$$

with $A(x, y) = -\delta(x - y)\partial^2$. To calculate the 2-point function we use the Green's function:

$$K(x, y) = \langle \mathrm{T}\phi(x)\phi(y)\rangle = A(x, y)^{-1},$$

where $K(x, y) = K(x - y)$ is the solution of the differential equation

$$\begin{aligned} -\partial^2 K(x) &= i\delta(x) \\ \int d^d x \; k^2 \tilde{K}(k) e^{-ik\cdot x} &= i \int d^d x \; e^{-ik\cdot x} \\ \Longrightarrow \tilde{K}(k) &= \frac{i}{k^2} \end{aligned}$$

So that

$$K(x) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + i\varepsilon} e^{ik\cdot x} \sim \frac{1}{|x|^{d-2}}.$$

Which is what we expected, since we know that this theory is conformal. Therefore its 2-point function should behave as $K(x) \sim 1/|x|^{2\Delta_\phi}$, and we know that the scaling dimension of the field ϕ is $\Delta_\phi = \frac{d-2}{2}$ in any dimension d.

7.2 Toy model for AdS/CFT

Let's consider an AdS_2 space, with the following (Poincaré) metric

$$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2}.$$

Suppose that we have a very massive particle in this background, then if we want to study what is the minimal path that a particle follows from a point x_1 to a point x_2 in the $x-$axis. Then, following the Feynman idea, we take the 2-point function as the sum of all probabilities of the paths that the particle can take to go from one point to the other, namely,

$$G_2(x_1, x_2) = \sum e^{-S(x_1,x_2)},$$

with the action being the one that describes relativistic particles of mass m, $S = m \int_{t_1}^{\tau_2} d\tau \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$. From the metric, it is clear that the particle will avoid going from the boundary $z = 0$, because it's very costly (it has to travel a distance proportional to $1/z^2$), so it will prefer to go through the "bulk". Since, the mass m is very large, then the most likely path will be governed by the one that minimizes the action (geodesic or classical solution), so that

$$G_2(x_1, x_2) \sim e^{-m \cdot L}, \qquad L = \text{minimal length.}$$

One can show that geodesics in AdS_2 are given by semi-circles, as shown in Figure 4,

$$z(x)^2 = R^2 - (x - x_1 - R)^2, \tag{7.2}$$

where here we can identify $x_2 = x_1 + 2R$.

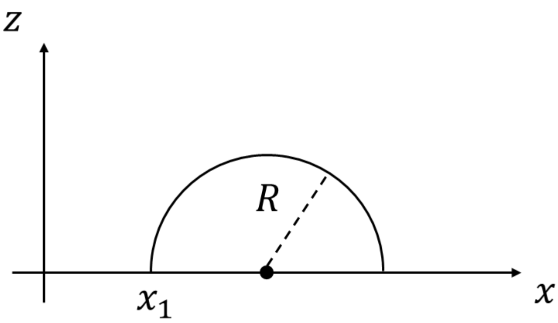


Figure 4. AdS_2 geodesics are semi-circles.

That is, (7.2) solves the equations of motion described by the Nambu-Goto action (in this case, this are just fancy words to say the action that describes the length of a free particle in a specific background):

$$S_{NG} = \int d\tau \sqrt{g_{\mu\nu}\dot{x}_\mu \dot{x}_\nu}. \tag{7.3}$$

One can show that this action is reparametrization invariant, i.e. invariant under the transformation $\tau \to f(\tau)$ (we can start to suspect from here that 1d CFTs have reparametrization invariance), therefore we can choose $x = \tau$ and the classical solutions of $S_{NG} = \int dx \sqrt{\frac{\dot{z}^2+1}{z^2}}$ are the semicircles described in (7.2).

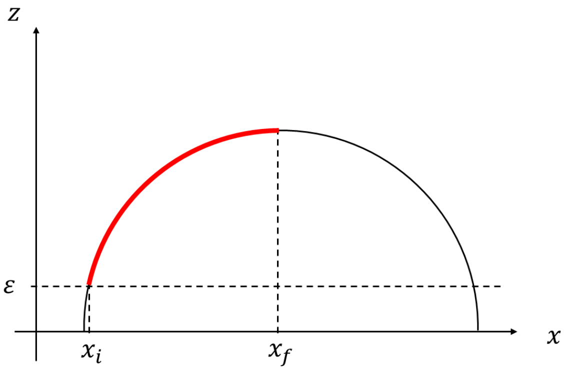


Figure 5. AdS$_2$ geodesics are semi-circles.

Let's consider half of the semi-circle (the full length is twice this result), and to avoid the $1/z$ divergene as $z \to 0$, we take $z_i = \varepsilon$. So, we want to calculate the length of the arc shown in Figure 5

The initial and final points are $x_i = x_1 + R - \sqrt{R^2 - \varepsilon^2} \approx x_1 + \varepsilon^2/(2R)$ and $x_f = x_1 + R$, respectively. So, the length of the arc is:

$$
\begin{aligned}
\ell &= \int_{x_i}^{x_f} dx \sqrt{\frac{\dot{z}^2 + 1}{z^2}} = \int_{x_i}^{x_f} dx \frac{R}{R^2 - (x - x_1 - R)^2} \\
&= \operatorname{arctanh}\left(\frac{x - x_1 - R}{R}\right)\Bigg|_{x_i}^{x_f} \\
&= \frac{1}{2}\ln\left(\frac{x_f - x_1}{x_f - x_1 - 2R} \cdot \frac{x_i - x_1 - 2R}{x_i - x_1}\right) \\
&= \frac{1}{2}\ln\left[\left(\frac{2R}{\varepsilon}\right)^2 - 1\right] \\
&\approx \ln\left(\frac{2R}{\varepsilon}\right).
\end{aligned}
\tag{7.4}
$$

Hence,

$$
G_2(R) = e^{-m \cdot 2\ell} = e^{-2m \ln\left(\frac{2R}{\varepsilon}\right)} = \left(\frac{\varepsilon}{2R}\right)^{2m}.
\tag{7.5}
$$

Now, what was the 2-point function of a 1d CFT with points? It is

$$
\langle \mathcal{O}(x_2)\mathcal{O}(x_1)\rangle = \frac{1}{|2(x_f - x_i)|^{2\Delta}} = \frac{1}{|2(R - \varepsilon^2/(2R))|^{2\Delta}} \approx \frac{1}{|2R|^{2\Delta}},
\tag{7.6}
$$

we can see that both correlation functions are the same with $\Delta = m$, up to this ε factor. But, that's fine because such finite result correspond to the renormalized operators $\mathcal{O}$. If we work with bare operators, say $\mathcal{O}_b$, then by multiplicative renormalization $\mathcal{O}(x) = \varepsilon^{-\Delta}\mathcal{O}_b(x)$ and

$$\langle \mathcal{O}_b(x)\mathcal{O}_b(x+R)\rangle = \left(\frac{\varepsilon}{2R}\right)^{2\Delta}.$$

So, the equilavence of the 2-point functions is now clear. We have started from AdS very massive dynamics in 2d and it generated an effective boundary behavior described by a 1d CFT.

What happens if we consider 3 very massive particles that meet at a point P in AdS, as shown in Figure 6? We do the same exercise: integrate over all posible trajectories that meet at P and then find the point P that minimizes the free action $S = m_1\ell_1 + m_2\ell_2 + m_3\ell_3$, where ℓ_α are the lengths of the arcs with initial points x_α, final point x_P, and radius $R_\alpha = \frac{(x_\alpha - x_P)^2 + z_P^2}{2|x_P - x_\alpha|}$, for $\alpha = 1, 2, 3$.

One can show that,

$$\ell_\alpha = \ln\frac{(x_\alpha - x_P)^2 + z_P^2}{\varepsilon z_P}. \tag{7.7}$$

And by solving $\partial_{x_P} S = \partial_{x_P} S = 0$, then we find

$$G_3(x_1, x_2, x_3) = \frac{C(m_1, m_2, m_3)}{|x_1 - x_2|^{m_1+m_2-m_3}|x_1 - x_3|^{m_1+m_3-m_2}|x_2 - x_3|^{m_2+m_3-m_1}}, \tag{7.8}$$

which resembles the 3-point function of a CFT with $\Delta_\alpha = m_\alpha$. Note that the structure constants in this case are completely determined by the masses, which are the only parameters in our system. However, in an interacting theory in AdS, the C's must depend on the coupling constants as well.

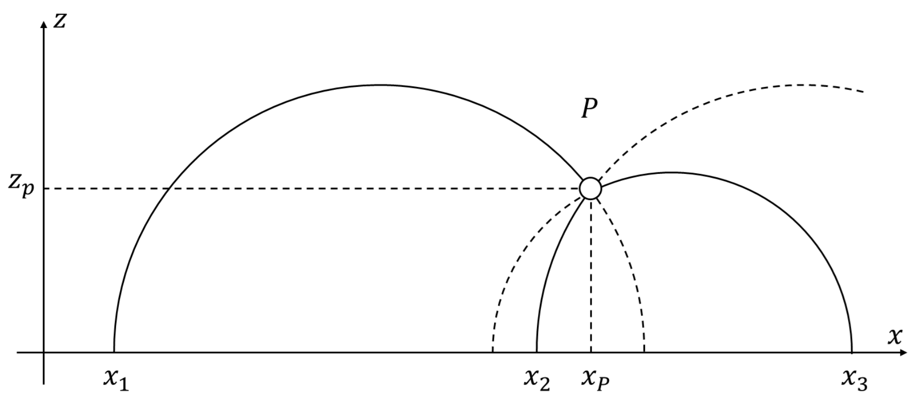


Figure 6. AdS_2 geodesics are semi-circles.

So, we can see that interactions of very massive particles in the AdS bulk, correspond to CFT correlation functions in the boundary. On the other hand, if we start from the CFT side, then correlation functions of operators with large scaling dimensions in the boundary correspond to massive particles moving through geodesics and interacting in the AdS bulk.

For completeness, what if we consider 2 particles that meet at the point P, should we get anything interesting? The answer is yes, if we follow the same exercise, we find that if the masses are different, then the minimization procedure has no solution, therefore there is no chance that this process happens (CFT 2-point function is zero in this case). But if the masses are the same, then we arrive at the same CFT 2-point function obtained before. Therefore,

$$G_2(x,y) = \frac{\delta_{m_1 m_2}}{|x-y|^{2m_1}},$$

as expected from the CFT side for primary operators with scaling dimensions m_1 and m_2.

7.3 The free boson in 2d

This is one of the simplest CFTs, with the following action

$$S = \frac{1}{2}\int d^2x\ \partial_\mu\phi\partial^\mu\phi. \tag{7.9}$$

Following the same procedure as before (in Euclidean signature), we want to calculate $K(x,y) = \langle\phi(x)\phi(y)\rangle = K(|x-y|) = K(r)$. So, in polar coordinates, we have the equation

$$\begin{aligned}
\delta(x,y) &= -\delta^2 K(r) \\
1 &= 2\pi\int_0^r d\rho\ \rho\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho K'(\rho)\right)\right] \\
\implies\ K'(r) &= -\frac{1}{2\pi r}.
\end{aligned}$$

Thus,

$$K(|x-y|) = \langle\phi(x)\phi(y)\rangle = -\frac{1}{4\pi}\ln(|x-y|^2) + \text{constant}.$$

If we take $z = x_1 + ix_2$ and $w = y_1 + iy_2$, then

$$\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\frac{1}{4\pi}\left[\ln(z-w) + \ln(\bar{z}-\bar{w})\right] + \text{constant}.$$

This may look bizarre, since we know that our theory is a CFT, why does the 2-point function have this form? The answer is because ϕ are not quite the primary fields (their conformal dimension is 0). But, if we consider the derivative of the previous correlation, we get,

$$\partial_z\partial_w\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\frac{1}{4\pi}\frac{1}{(z-w)^2},$$

and similarly for $\partial_{\bar{z}}\partial_{\partial w}$, so

$$\langle \partial_z \phi(z,\bar{z}) \partial_w \phi(w,\bar{w}) \rangle = -\frac{1}{4\pi} \frac{1}{(z-w)^2} \qquad \text{(holomorphic)}$$
$$\langle \partial_{\bar{z}} \phi(z,\bar{z}) \partial_{\bar{w}} \phi(w,\bar{w}) \rangle = -\frac{1}{4\pi} \frac{1}{(z-w)^2} \qquad \text{(anti-holomorphic)} \tag{7.10}$$

Therefore, the OPE on the holomorphic part reads:

$$\partial\phi(z)\partial\phi(w) \sim -\frac{1}{4\pi}\frac{1}{(z-w)^2}, \tag{7.11}$$

where $\partial\phi(z) = \partial_z\phi(z)$. Note that the scaling dimension of $partial\phi$ is $\Delta_{partial\phi} = 1$ and since this is a scalar field it has zero spin, so that $h = 1$.

The stress-energy tensor for this system is $T_{\mu\nu} = \left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right)$, which in $\{z, \bar{z}\}$ coordinates has the form:

$$T(z) = -2\pi T_{zz} = -2\pi :\partial\phi\partial\phi: = -2\pi \lim_{w\to z}\left(\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle\right) \tag{7.12}$$

the mixed terms vanish, as can be shown by using the metric form in (5.7). And we have used normal ordering $:\ \cdot\ :$ to ensure the vanishing of the vacuum expectation value. We then proceed to calculate the OPE of $T(z)$ with $\partial\phi$, as follows

$$\begin{aligned}
T(z)\partial\phi(w) &= -2\pi \lim_{z\to z'}\left(\partial\phi(z)\partial\phi(z') - \langle\partial\phi(z)\partial\phi(z')\rangle\right)\partial\phi(w) \\
&\underset{\text{(Wick's theorem)}}{\sim} -4\pi\,\langle\partial\phi(z)\partial\phi(w)\rangle\,\partial\phi(z) \\
&= -4\pi\left(-\frac{1}{4\pi}\right)\frac{\partial\phi(z)}{(z-w)^2} \\
&\underset{\text{(expand around } z=w)}{\sim} \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)}.
\end{aligned}$$

From (6.4), we can see that $h_{\partial\phi} = 1$, as expected and thus, is a primary field. If we now calculate the OPE of the stress-energy tensor with itself, we arrive

$$\begin{aligned}
T(z)T(w) &= 4\pi^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&= 4\pi^2 \lim_{z\to z'}\left(\partial\phi(z)\partial\phi(z') - \langle\partial\phi(z)\partial\phi(z')\rangle\right)\lim_{w\to w'}\left(\partial\phi(w)\partial\phi(w') - \langle\partial\phi(w)\partial\phi(w')\rangle\right) \\
&\underset{\text{(Wick's theorem)}}{\sim} 4\pi^2(2)\,\langle\partial\phi(z)\partial\phi(w)\rangle^2 + 4\pi^2(4)\,\langle\partial\phi(z)\partial\phi(w)\rangle\,\partial\phi(z)\partial\phi(w) \\
&= \frac{1/2}{(z-w)^4} - 4\pi\frac{\partial\phi(z)\partial\phi(w)}{(z-w)^2} \\
&\underset{\text{(expand around } z=w)}{\sim} \frac{1/2}{(z-w)^4} - \frac{4\pi:\partial\phi(w)\partial\phi(w)}{(z-w)^2} - \frac{4\pi(1/2)\partial_w:\partial\phi(w)\partial\phi(w):}{(z-w)} \\
&= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}.
\end{aligned} \tag{7.13}$$

We can see, comparing to (6.4), that T is not a primary operator because of the presence of an anomalous term $\frac{1/2}{(z-w)^4}$, compared to the $d \neq 2$ case, where the stress-energy tensor is a (quasi) primary operator. Though, we can read off its conformal dimension from the second term, which gives us $h_T = 2$ (this matches our expectation that $T_{\mu\nu}$ is a spin 2 field with scaling dimension $\Delta_T = 2$).

7.4 The free fermion in 2d

The free fermion action

$$S = \frac{1}{2} \int d^2x \; \bar{\Psi} \gamma^\mu \partial_\mu \Psi, \tag{7.14}$$

with $\gamma^0 = \sigma^1$, $\gamma^1 = \sigma^2$, and $\gamma^0\gamma^1 = i\sigma^3$. If we expand the term:

$$\gamma^0 \gamma^\mu \partial_\mu = \partial_0 + i\sigma^3 \partial_1 = \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix} = 2 \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix}.$$

And then, by considering $\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$, the action reads

$$S = \int d^2x \; \left(\bar{\psi} \partial \bar{\psi} + \psi \bar{\partial} \psi \right) \tag{7.15}$$

with equations of motion

$$\begin{aligned} \bar{\partial}\psi &= 0, \qquad &&\text{(holomorphic solution)} \\ \partial\bar{\psi} &= 0 \qquad &&\text{(anti-holomorphic solution)} \end{aligned} \tag{7.16}$$

By a similar procedure as the previous section, we can calculate the 2-point functions

$$\begin{aligned} \langle \psi(z)\psi(w) \rangle &= \frac{1}{2\pi} \frac{1}{z-w}, \qquad &&\text{(holomorphic)} \\ \langle \bar{\psi}(z)\bar{\psi}(w) \rangle &= \frac{1}{2\pi} \frac{1}{\bar{z}-\bar{w}}, \qquad &&\text{(anti-holomorphic)} \\ \langle \psi(z)\psi(\bar{w}) \rangle &= 0. \end{aligned} \tag{7.17}$$

So that, we can read the scaling dimensions of the primary fields ψ and $\bar{\psi}$: $\Delta_\psi = \Delta_{\bar{\psi}} = 1/2$. Also, we can get

$$\begin{aligned} \langle \partial_z \psi(z) \psi(w) \rangle &= -\frac{1}{2\pi} \frac{1}{(z-w)^2}, \\ \langle \partial_z \psi(z) \partial_w \psi(w) \rangle &= -\frac{1}{\pi} \frac{1}{(z-w)^3} \\ T^{\bar{z}\bar{z}} &= 2\psi\partial\psi \\ &\vdots \end{aligned} \tag{7.18}$$

The renormalized holomorphic energy-momentum tensor is $T(z) = -\pi : \psi(z)\partial\psi(z) :$. So, the OPE expansion of T with ψ is

$$
\begin{aligned}
T(z)\psi(w) &= -\pi : \psi(z)\partial\psi(z) : \psi(w) \\
\text{(Wick's theorem)} \quad &\sim \pi \langle \psi(z)\psi(w) \rangle \partial\psi(z) - \pi \langle \partial\psi(z)\psi(w) \rangle \psi(z) \\
&\sim \frac{1}{2}\frac{\partial\psi(z)}{z-w} + \frac{1}{2}\frac{\psi(z)}{(z-w)^2} \\
\text{(expand around z=w)} \quad &\sim \frac{1/2\ \psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{(z-w)}
\end{aligned} \tag{7.19}
$$

Comparing with (6.4), we can see that $h_\psi = 1/2$, as expected for a spin 1/2 field with scaling dimension 1/2. This is thus, a primary field.

If we calculate the OPE of T with itself, we get

$$
\begin{aligned}
T(z)T(w) &= \pi^2 : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \\
&\sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)},
\end{aligned} \tag{7.20}
$$

and from (6.4), we conclude that T is not a primary field, but has conformal dimension $h_T = 2$. The anomalous term is called *central charge* and is non-universal.

8 The central charge

The general OPE of the (renormalized) stress-energy tensor with itself has the form:

$$
T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \tag{8.1}
$$

where c is the model-dependent central charge, determined by the short distance behavior of the theory. In the case of the free-boson and the free-fermion the values of c are 1 and 1/2, respectively.

Now that we know that T is not a primary field, then we should ask ourselves: how does T transform?

For this, we can use that the infinitesimal variation of a primary field (or chain of primaries) $\mathcal{O}$ under conformal transformations is[9]:

$$
\delta_\epsilon \mathcal{O}(w) = -\frac{1}{2\pi i}\oint_C dz\ \epsilon(z)T(z)\mathcal{O}(w), \tag{8.2}
$$

replacing $\mathcal{O}$ by T, we get

[9] The derivation of this is shown step by step in chapter 5.2.2 of [1].

$$\delta_\epsilon T(w) = -\frac{1}{2\pi i}\oint_C dz\ \epsilon(z)T(z)T(w) \tag{8.3}$$

$$= -\frac{c}{12}\partial_w^3\epsilon(w) - 2T(w)\partial_w\epsilon(w) - \epsilon(w)\partial_w T(w). \tag{8.4}$$

To know the full action of this transformation, we have to "exponentiate" the previous generator and we obtain that under $z \to w(z)$

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right]$$

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) + \frac{c}{12}\{z;w\}, \tag{8.5}$$

where $\{w;z\} = \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2$ is the Schwarzian derivative of w evaluated at z. And in the second line, we used the identity:

$$\{u;z\} = \left(\frac{dw}{dz}\right)^2\{u;w\} + \{w;z\}$$

By looking at (**??**), we see that T is not a primary operator because of the presence of c. However, this should be a consequence of local group because we know that for the global conformal group, T is a (quasi-)primary operator. So, if we focus on the $SL(2,\mathbb{C})$ part, i.e. the transformations of the form

$$w(z) = \frac{az+b}{cz+d}, \qquad ad - bc = 1$$

then we can show that they have a vanishing Schwarzian derivative. Which is what we expected. So, we are safe!

9 More on c

Last time, we briefly touched upon one of the interpretations of the central charge c. Today, we will make this more precise by showing how the free energy of a plane and in a cylinder can be related to each other by means of the central charge. We will be following [18]. We consider a cylinder, which can be parametrised by the complex coordinate $w = \sigma + i\tau$ with $\sigma \sim \sigma + 2\pi$ which ensures the periodicity of the cylinder. We can map this to the plane as $z = e^{-iw}$ so that the σ coordinate now parametrises the circles, in the plane, and τ the size of these. Since this transformation is holomorphic, it is conformal, and we can go on to calculate how the stress energy tensors are related using 8.5. One can write

$$T(z)_{plane} = -z^{-2}\left(T(w)_{cylinder} - \frac{c}{12}S(z,w)\right) \tag{9.1}$$

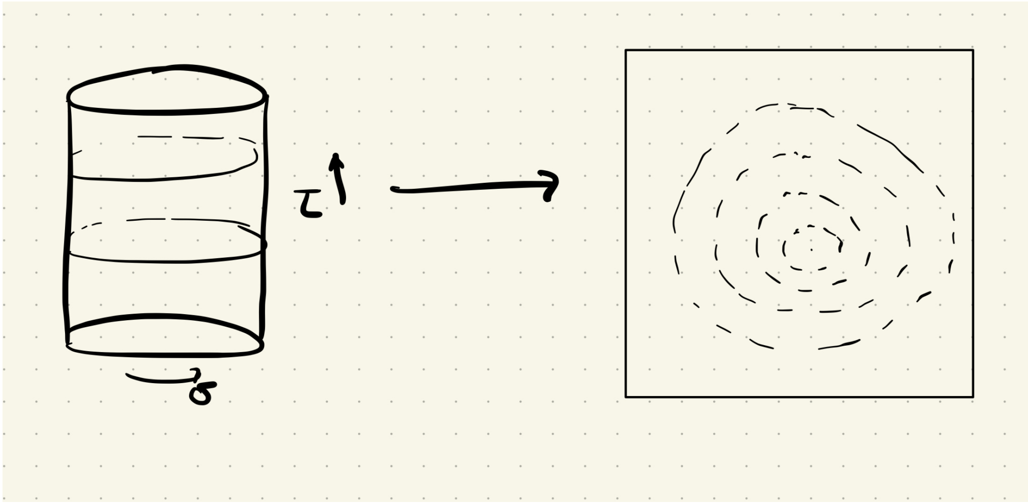


Figure 7. Conformal mapping from cylinder to plane

now

$$S(z,w) = \frac{(-i)^3 z}{(-iz)} - \frac{3}{2}\frac{((-i)^2 z)^2}{(-iz)^2} = -1 + 3/2 = 1/2 \tag{9.2}$$

And so we can write

$$T(w)_{cylinder} = -z^2 T(z)_{plane} + \frac{c}{24} \tag{9.3}$$

Now consider a theory with vanishing ground state energy on the plane: $\langle T_{plane}\rangle_{=0}$. We can see what this means in the cylinder by using

$$H = \int d\sigma T_{\tau\tau} = -\int d\sigma (T_{ww} + \bar{T}_{\bar{w}\bar{w}}) \tag{9.4}$$

And so $E = \frac{-2\pi(c+\bar{c})}{24}$ with c=1 for a free scalar, so that the energy density is -12

10 Radial Quantization

Staying with our friend the cylinder states live at constant σ and are evolved by $H = \partial_\tau$. In the plane, the dilation becomes $D = z\partial + \bar{z}\bar{\partial}$. The states now live at constant radius, and their evolution operator is the Dilaton. This is called radial quantization as now we have a radial ordering operator as opposed to time ordering. Clearly, sending $\tau \to -\infty$ corresponds to $z \to 0$. We can now assume the existance of a vacuum state $|0\rangle$, and that in the infinite past, the field is free, so we can write $|\phi_{\text{in}}\,\rangle = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0\rangle$. In Euclidean spacetime, the coordinate $\tau = it$ must flip sign under a hermitian conjugation so that t is left unchanged. Thus, we make the definition $[\phi(z,\bar{z})]^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\phi(1/\bar{z},1/z)$ where ϕ is a quasiprimary field. The powers of h and $\bar{h}$ come from the fact that we want a normalisable state when contracting it with the in state;

$$\langle \phi_{\text{out}} \mid \phi_{\text{in}} \rangle = \lim_{z,\bar{z},w,\bar{w}\to 0} \langle 0|\phi(z,\bar{z})^\dagger \phi(w,\bar{w})|0\rangle \tag{10.1}$$

$$= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0|\phi(1/\bar{z},1/z)\phi(w,\bar{w})|0\rangle \tag{10.2}$$

$$= \lim_{\xi,\bar{\xi}\to\infty} \bar{\xi}^{2h}\xi^{2\bar{h}} \langle 0|\phi(\bar{\xi},\xi)\phi(0,0)|0\rangle \tag{10.3}$$

so the last equation is independent of ξ because of the form of the two point function of quasiprimary operators.

We may expand conformal field $\phi(z,\bar{z})$ of dimensions $(h,\bar{h})$ as follows:

$$\phi(z,\bar{z}) = \sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n} \tag{10.4}$$

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz z^{m+h-1}\frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+\bar{h}-1}\phi(z,\bar{z}) \tag{10.5}$$

conjugating,

$$\phi(z,\bar{z})^\dagger = \sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} \bar{z}^{-m-h}z^{-n-\bar{h}}\phi^\dagger_{m,n} \tag{10.6}$$

while the previous definition gives

$$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\phi(1/\bar{z},1/z) \tag{10.7}$$

$$= \bar{z}^{-2h}z^{-2\bar{h}}\sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}}\phi_{m,n}\bar{z}^{m+h}z^{n+\bar{h}} \tag{10.8}$$

$$= \sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}}\phi_{-m,-n}\bar{z}^{-m-h}z^{-n-\bar{h}} \tag{10.9}$$

And so we require

$$\phi^\dagger_{m,n} = \phi_{-m,-n}$$

For the in and out states to be well defined, we require $\phi_{m,n}|0\rangle = 0 \quad (m > -h, n > -\bar{h})$. Henceforth, we will use only the holomorphic part of the fields to ease the notation. The radial ordering of fields is such that

$$\mathcal{R}\Phi_1(z)\Phi_2(w) = \begin{cases} \Phi_1(z)\Phi_2(w) & \text{if} \quad |z| > |w| \\ \Phi_2(w)\Phi_1(z) & \text{if} \quad |z| < |w| \end{cases} \tag{10.10}$$

For $a(z)$ and $b(w)$ holomorphic, we consider integrating clockwise around z:

$$\oint_w dza(z)b(w) \tag{10.11}$$

We can deform the contours as follows: consider two circles of radii $w \pm \epsilon$. Since we have not crossed any poles, this integral will be equivalent to that of the contour around w.

$$\begin{aligned}\oint_w dza(z)b(w) &= \oint_{C_1} dza(z)b(w) - \oint_{C_2} dzb(w)a(z) \\ &= [A, b(w)]\end{aligned} \tag{10.12}$$

with

$$A = \oint a(z)dz \tag{10.13}$$

Then for two operators that can be written as holomorphic densities, we find

$$[A, B] = \oint_0 dw \oint_w dz a(z)b(w) \tag{10.14}$$

allowing us to relate the OPE into operator language.

10.1 Virasoro algebra

We had seen that (looking only at the holomorphic part as usual)

$$\delta_\epsilon \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \langle T(z) X \rangle \tag{10.15}$$

if one defines

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz \epsilon(z) T(z) \tag{10.16}$$

it is easy to see, that using the previous results, we can rewrite this as

$$\delta_\epsilon \Phi(w) = -\left[Q_\epsilon, \Phi(w)\right] \tag{10.17}$$

and so Q_ϵ generates conformal transformations. Consider the expansion of the holomorphic stress energy tensor, which has conformal weight 2:

$$T(z) = \sum_{n \in \mathbf{Z}} z^{-n-2} L_n \quad L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \tag{10.18}$$

we can also expand the infinitessimal coordinate change as

$$\epsilon(z) = \sum_{n \in \mathbf{Z}} z^{n+1} \epsilon_n \tag{10.19}$$

so that

$$Q_\epsilon = \sum_{n \in \mathbf{Z}} \epsilon_n L_n \tag{10.20}$$

meaning that the mode operators L_n generate the local conformal transforms on Hilbert space, thus drawing an equivalence with the generators of conformal mappings of space.

Using the residue theorem for higher order poles

$$\mathrm{Res}(f, c) = \frac{1}{(n-1)!} \lim_{z \to c} \frac{d^{n-1}}{dz^{n-1}} \left[(z-c)^n f(z)\right]$$

it is the straightforward to compute

$$[L_n, L_m] = \frac{1}{(2\pi i)^2}\oint_0 dw w^{m+1}\oint_w dz z^{n+1}\left\{\frac{c/2}{(z-w)^4}\right. \tag{10.21}$$

$$\left. + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{ reg. }\right\} \tag{10.22}$$

$$= \frac{1}{2\pi i}\oint_0 dw w^{m+1}\left\{\frac{1}{12}c(n+1)n(n-1)w^{n-2} + \right. \tag{10.23}$$

$$\left. 2(n+1)w^n T(w) + w^{n+1}\partial T(w)\right\} \tag{10.24}$$

$$= \frac{1}{12}cn\left(n^2-1\right)\delta_{n+m,0} + 2(n+1)L_{m+n} \tag{10.25}$$

$$- \frac{1}{2\pi i}\oint_0 dw (n+m+2) w^{n+m+1} T(w) \tag{10.26}$$

$$= \frac{1}{12}cn\left(n^2-1\right)\delta_{n+m,0} + (n-m)L_{m+n} \tag{10.27}$$

10.2 Hilbert Space

We know that the vacuum of a theory should be invariant under the (global) symmetries of the theory, and so in this case we should demand L_{-1}, L_0, L_1 should yield zero when acting on the vacuum. This is because the vacuum should preserve the global structure of the theory, but local generators can create excited states as we will see. The invariance of the vacuum under global transformations is also a requirement for the uniqueness, which we would need if we are in a phase with no SSB. Since the Hamiltonian is $L_0 + \bar{L}_0$, we also find that this implies that the energy of the vacuum is zero. Looking at the definition of T 10.18 and requiring it to be finite at $z = 0$, we require

$$\begin{aligned} L_n|0\rangle &= 0 \\ \bar{L}_n|0\rangle &= 0 \end{aligned} \quad (n \geq -1) \tag{10.28}$$

Meaning that the VEV of the stress energy tensor is zero $\langle 0|T(z)|0\rangle = \langle 0|\bar{T}(\bar{z})|0\rangle = 0$. The way we will build our hilbert space is by acting on the vacuum with L_n as we will see. We can figure out the action of L_n on a field as

$$[L_n, \phi(w,\bar{w})] = \frac{1}{2\pi i}\oint_w dz z^{n+1} T(z)\phi(w,\bar{w}) \tag{10.29}$$

$$= \frac{1}{2\pi i}\oint_w dz z^{n+1}\left[\frac{h\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{z-w} + \text{ reg. }\right] \tag{10.30}$$

$$= h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \quad (n \geq -1) \tag{10.31}$$

we can now consider a state with weight $h, \bar{h}$: $|h, \bar{h}\rangle \equiv \phi(0,0)|0\rangle$, and clearly, $L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle$ and similarly for $\bar{L}_0$. This means that $\bar{h}\rangle$ is an eigenstate of the hamiltonian. We also have ladder opperators with $n > 0$ anihilating this state.

$$[L_n, \phi_m] = [n(h-1) - m]\phi_{n+m} \tag{10.32}$$

in particular,

$$[L_0, \phi_m] = -m\phi_m \tag{10.33}$$

and so ϕ_m acts as raising and lowering operators for eigenstates of L_0. Similarly, we can also use L_{-m} as operators that increase the conformal dimension of a state by m: $[L_0, L_{-m}] = mL_{-m}$ we can then apply successive iterations of the generators to create states with different h $L_{-k_1}L_{-k_2}\cdots L_{-k_n}|h\rangle$. where $h' = h + k_1 + k_2 + \cdots + k_n \equiv h + N$. These states are descendant and they form a representation of the virasoro algebra as the subset of states generated by $\langle h\rangle$ is closed. These subspaces are labeled by h and are called Verma modules. N is called the level of the descendant, and there are p(N) at a given level N where p(N) is the partitions of N. We can have a generation function for this:

$$\frac{1}{\varphi(q)} \equiv \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n) q^n \tag{10.34}$$

10.3 Canonical quantisation of cylinder

As usual, we begin by considering the simplest examples and build from there. We want to quantise a free scalar on a cylinder with circumference L so that $\varphi(x+L,t) \equiv \varphi(x,t)$ and we can Fourier expand

$$\varphi(x,t) = \sum_n e^{2\pi i n x/L} \varphi_n(t) \tag{10.35}$$

$$\varphi_n(t) = \frac{1}{L}\int dx e^{-2\pi i n x/L} \varphi(x,t) \tag{10.36}$$

In terms of the Fourier coefficients φ_n, the free field Lagrangian

$$\frac{1}{2} g \int dx \left\{ (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right\} \tag{10.37}$$

becomes

$$\frac{1}{2} g L \sum_n \left\{ \dot{\varphi}_n \dot{\varphi}_{-n} - \left(\frac{2\pi n}{L}\right)^2 \varphi_n \varphi_{-n} \right\} \tag{10.38}$$

The momentum conjugate and commutation relations are

$$\pi_n = gL\dot{\varphi}_{-n} \quad [\varphi_n, \pi_m] = i\delta_{nm} \tag{10.39}$$

We make the creation/anihilation operators $\tilde{a}_n$ and $\tilde{a}_n^\dagger$ as usual

$$\tilde{a}_n = \frac{1}{\sqrt{4\pi g|n|}} \left(2\pi g|n|\varphi_n + i\pi_{-n}\right) \tag{10.40}$$

such that $[\tilde{a}_n, \tilde{a}_m] = 0$ and $\left[\tilde{a}_n, \tilde{a}_m^\dagger\right] = \delta_{mn}$; we will deal with φ_0 later as it evidently diverges. We can now normalise them as

$$a_n = \begin{cases} -i\sqrt{n}\tilde{a}_n & (n>0) \\ i\sqrt{-n}\tilde{a}_{-n}^\dagger & (n<0) \end{cases} \quad \bar{a}_n = \begin{cases} -i\sqrt{n}\tilde{a}_{-n} & (n>0) \\ i\sqrt{-n}\tilde{a}_n^\dagger & (n<0) \end{cases} \tag{10.41}$$

and treat the zero mode φ_0 separately. The associated commutation relations are

$$[a_n, a_m] = n\delta_{n+m} \qquad [a_n, \bar{a}_m] = 0 \qquad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m} \tag{10.42}$$

The Hamiltonian is then expressible as

$$H = \frac{1}{2gL}\pi_0^2 + \frac{\pi}{L}\sum_{n\neq 0}\left(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n\right) \tag{10.43}$$

The commutation relations lead to the relation

$$[H, a_{-m}] = \frac{2\pi}{L} m a_{-m} \tag{10.44}$$

which means that $a_{-m}(m > 0)$, when applied to an eigenstate of H of energy E, produces another eigenstate with energy $E + 2m\pi/L$.

Since the Fourier modes are

$$\varphi_n = \frac{i}{n\sqrt{4\pi g}}\left(a_n - \bar{a}_{-n}\right) \tag{10.45}$$

the mode expansion at $t = 0$ may be written as

$$\varphi(x) = \varphi_0 + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n - \bar{a}_{-n}\right)e^{2\pi i n x/L} \tag{10.46}$$

The time evolution of the operators φ_0, a_n, and $\bar{a}_n$ in the Heisenberg picture follows immediately from the above Hamiltonian:

$$\varphi_0(t) = \varphi_0(0) + \frac{1}{gL}\pi_0 t$$

$$a_n(t) = a_n(0)e^{-2\pi i n t/L} \tag{10.47}$$

$$\bar{a}_n(t) = \bar{a}_n(0)e^{-2\pi i n t/L} \tag{10.48}$$

In terms of constant operators, the mode expansion of the field at arbitrary time is then

$$\varphi(x,t) = \varphi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi i n(x-t)/L} - \bar{a}_{-n}e^{2\pi i n(x+t)/L}\right) \tag{10.49}$$

If we go over to Euclidean space-time (i.e., replace t by $-i\tau$) and use the conformal coordinates

$$z = e^{2\pi(\tau - ix)/L} \qquad \bar{z} = e^{2\pi(\tau+ix)/L} \tag{10.50}$$

we finally obtain the expansion of φ and its derivative as

$$\varphi(z,\bar{z}) = \varphi_0 - \frac{i}{4\pi g}\pi_0 \ln(z\bar{z}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n z^{-n} + \bar{a}_n\bar{z}^{-n}\right) \tag{10.51}$$

$$i\partial\varphi(z) = \frac{1}{4\pi g}\frac{\pi_0}{z} + \frac{1}{\sqrt{4\pi g}}\sum_{n\neq 0} a_n z^{-n-1} \tag{10.52}$$

We may introduce two operators a_0 and $\bar{a}_0$:

$$a_0 \equiv \bar{a}_0 \equiv \frac{\pi_0}{\sqrt{4\pi g}} \tag{10.53}$$

which allow us to include the zero-mode term into the sum:

$$i\partial\varphi(z) = \frac{1}{\sqrt{4\pi g}}\sum_{n} a_n z^{-n-1} \tag{10.54}$$

a_n id s creation operator of "right-moving" excitations, whereas the $\bar{a}_n$ are associated with "left-moving" excitations.

6.3.2. Vertex Operators

Since the canonical scaling dimension of the boson φ vanishes, it is possible to construct an infinite variety of local fields related to φ without introducing a scale, namely the so-called vertex operators:

$$\nu_\alpha(z,\bar{z}) =: e^{i\alpha\varphi(z,\bar{z})} : \tag{10.55}$$

The normal ordering has the following meaning, in terms of the operators appearing in the mode expansion:

$$\begin{aligned}\nu_\alpha(z,\bar{z}) = \exp&\left\{ i\alpha\varphi_0 + \frac{\alpha}{\sqrt{4\pi g}}\sum_{n>0}\frac{1}{n}\left(a_{-n}z^n + \bar{a}_{-n}\bar{z}^n\right)\right\} \\ \times \exp&\left\{\frac{\alpha}{4\pi g}\pi_0 - \frac{\alpha}{\sqrt{4\pi g}}\sum_{n>0}\frac{1}{n}\left(a_n z^{-n} + \bar{a}_n\bar{z}^{-n}\right)\right\}\end{aligned} \tag{10.56}$$

Within each exponential, the different operators commute.

We shall now demonstrate that these fields are primary, with holomorphic and anti-holomorphic dimensions

$$h(\alpha) = \bar{h}(\alpha) = \frac{\alpha^2}{8\pi g} \tag{10.57}$$

We first calculate the OPE of $\partial\varphi$ with V_α:

$$\begin{aligned}\partial\varphi(z)\mathcal{V}_\alpha(w,\bar{w}) &= \sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\partial\varphi(z) : \varphi(w,\bar{w})^n : \\ &\sim -\frac{1}{4\pi g}\frac{1}{z-w}\sum_{n=1}^{\infty}\frac{(i\alpha)^n}{(n-1)!} : \varphi(w,\bar{w})^{n-1} : \\ &\sim -\frac{i\alpha}{4\pi g}\frac{\nu_\alpha(w,\bar{w})}{z-w}\end{aligned} \tag{10.58}$$

Next, we calculate the OPE of V_α with the energy-momentum tensor:

$$\begin{aligned} T(z)\mathcal{V}_\alpha(w,\bar{w}) &= -2\pi g \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} : \partial\varphi(z)\partial\varphi(z) :: \varphi(w,\bar{w})^n : \\ &\sim -\frac{1}{8\pi g}\frac{1}{(z-w)^2}\sum_{n=2}^{\infty}\frac{(i\alpha)^n}{(n-2)!} : \varphi(w,\bar{w})^{n-2} : \\ &\quad + \frac{1}{z-w}\sum_{n=1}^{\infty}\frac{(i\alpha)^n}{n!} n : \partial\varphi(z)\varphi(w,\bar{w})^{n-1} : \\ &\sim \frac{\alpha^2}{8\pi g}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w \mathcal{V}_\alpha(w,\bar{w})}{z-w} \end{aligned} \tag{10.59}$$

To the n-th term in the summation we have applied $2n$ single contractions and $n(n-1)$ double contractions. We have replaced $\partial\varphi(z)$ by $\partial\varphi(w)$ in the last equation since the difference between the two leads to a regular term. It is now clear by the form of this OPE that ν_α is primary, with the conformal weight given above. The OPE with $\bar{T}$ has exactly the same form.

In order to calculate the OPE of products of vertex operators, we may use the following relation for a single harmonic oscillator:

$$: e^{A_1} :: e^{A_2} :=: e^{A_1+A_2} : e^{(A_1 A_2)} \tag{10.60}$$

where $A_i = \alpha_i a + \beta_i a^\dagger$ is some linear combination of annihilation and creation operators (this relation is demonstrated in App. 6.A). Since a free field is simply an assembly of decoupled harmonic oscillators, the same relation holds if A_1 and A_2 are linear functions of a free field. In particular, we may write

$$: e^{a\varphi_1} :: e^{b\varphi_2} :=: e^{a\varphi_1+b\varphi_2} : e^{ab\langle\varphi_1\varphi_2\rangle} \tag{10.61}$$

Applied to vertex operators, this relation yields

$$\nu_\alpha(z,\bar{z})\mathcal{V}_\beta(w,\bar{w}) \sim |z-w|^{2\alpha\beta/4\pi g}\nu_{\alpha+\beta}(w,\bar{w}) + \cdots \tag{10.62}$$

However, we have seen previously that invariance under the global conformal group forces the fields within a nonzero two-point function to have the same conformal dimension. Furthermore, the requirement that the correlation function $\langle\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_\beta(w,\bar{w})\rangle$ does not grow with distance imposes the constraint $\alpha\beta < 0$, which leaves $\alpha = -\beta$ as the only possibility ($g = 1/4\pi$):

$$\nu_\alpha(z,\bar{z})\mathcal{V}_{-\alpha}(w,\bar{w}) \sim |z-w|^{-2\alpha^2} + \cdots \tag{10.63}$$

In general, the correlator of a string of vertex operators ν_{α_i} vanishes unless the sum of the charges vanishes: $\sum_i \alpha_i = 0$; this will be demonstrated in Chap. 9, in which vertex operators will be further studied. From now on, the normal ordering of the vertex operator will not be explicitly written but will always be implicit.

11 Some more examples

11.1 The Compactified Boson

We already study the free boson at the classical level. One can notice easily that the scalar is invariant under shift symmetry: $\phi \to \phi+\text{const}$. So, we can restrict ourselves to the domain of a circle of radius R, so that $\phi \sim \phi + 2\pi R$. The modifications from our previous discussion are:

1. The center of mass momentum π_0 must take values that are integer multiples of $1/R$ in order for $\mathcal{V}_\alpha$ to be well-defined.
2. Boundary condition: when we circle once the cylinder, then φ winds around its field configuration m times, i.e. $\varphi(x+L,t) = \varphi(x,t) + 2\pi m R$.

So, our field reads

$$\varphi_{m,n}(x,t) = \varphi_0 + \frac{n}{4\pi g L}t + \frac{2\pi R m}{L}x + \frac{i}{\sqrt{4\pi g}}\sum_{k\neq 0}\frac{1}{k}\left(a_k e^{2\pi i k(x-t)/L} - \bar{a}_{-k}e^{2\pi i k(x+t)/L}\right), \tag{11.1}$$

from which, as before, we can get the holomorphic part of the operator $\partial\varphi(z)$

$$i\partial\varphi(z) = \left(\frac{n}{4\pi g R} + \frac{mR}{2}\right)\frac{1}{z} + \frac{1}{\sqrt{4\pi g}}\sum_{k\neq 0} a_k z^{-k-1} \tag{11.2}$$

So, letting $a_0 = \sqrt{4\pi g}\left(\frac{n}{4\pi g R} + \frac{mR}{2}\right)$ and $\bar{a}_0 = \sqrt{4\pi g}\left(\frac{n}{4\pi g R} - \frac{mR}{2}\right)$, we can write

$$i\partial\varphi(z) = \frac{1}{\sqrt{4\pi g}}\sum_k a_k z^{-k-1}, \tag{11.3}$$

this is our Laurent expansion for the primary field $\partial\varphi$.
Then, the stress energy tensor is

$$T(z) = -2\pi g : \partial\varphi(z)\partial\varphi(z) := \frac{1}{2}\sum_{n,m\in\mathbb{Z}} z^{-n-m-2} : a_n a_m := \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \tag{11.4}$$

where in the last line we just write the Laurent expansion of T. Equating the last two lines gives us (similar expression for the anti-holomorphic part)

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_{n-m} a_m : \qquad (n\neq 0) \tag{11.5}$$

$$L_0 = \sum_{m>0} a_{-m} a_m + \frac{1}{2}a_0^2. \tag{11.6}$$

Therefore, recalling

$$H = \frac{4\pi g}{2gL} a_0^2 + \frac{2\pi}{L} \sum_{n>0} (a_{-n} a_n + \bar{a}_{-n} \bar{a}_n) = \frac{2\pi}{L} \left[a_0^2 + \sum_{n>0} (a_{-n} a_n + \bar{a}_{-n} \bar{a}_n) \right] \tag{11.7}$$

and from the previous equation,

$$\boxed{H = \frac{2\pi}{L} \left(L_0 + \bar{L}_0 \right)} \tag{11.8}$$

Specializing in our case, we have

$$L_0 = \sum_{k>0} a_{-k} a_k + 2\pi g \left(\frac{n}{4\pi g R} + \frac{mR}{2} \right)^2 \tag{11.9}$$

$$\bar{L}_0 = \sum_{k>0} \bar{a}_{-k} \bar{a}_k + 2\pi g \left(\frac{n}{4\pi g R} - \frac{mR}{2} \right)^2 \tag{11.10}$$

$$H = \frac{2\pi}{L} \sum_{k>0} \left(a_{-k} a_k + \bar{a}_{-k} \bar{a}_k + \frac{n^2}{4\pi g R^2} + m^2 R^2 \pi g \right). \tag{11.11}$$

If we label our vacuum as $|m, n\rangle$, so that $a_k |m, n\rangle = \bar{a}_k |m, n\rangle = 0$ for $k > 0$, then

$$L_0 |m, n\rangle = 2\pi g \left(\frac{n}{4\pi g R} + \frac{mR}{2} \right)^2 |m, n\rangle = h_{m,n} |m, n\rangle, \tag{11.12}$$

$$\bar{L}_0 |m, n\rangle = 2\pi g \left(\frac{n}{4\pi g R} - \frac{mR}{2} \right)^2 |m, n\rangle = \bar{h}_{m,n} |m, n\rangle, \tag{11.13}$$

$$(L_0 + \bar{L}_0) |m, n\rangle = \frac{n^2}{4\pi g R^2} + m^2 R^2 \pi g = \Delta_{m,n} |m, n\rangle, \tag{11.14}$$

$$(L_0 - \bar{L}_0) |m, n\rangle = nm = s_{m,n} |m, n\rangle, \tag{11.15}$$

from where we can read out the conformal dimension, the scaling dimension, and the spin of the vacuum states, respectively.

12 Some more examples

12.1 The Compactified Boson

We already study the free boson at the classical level. One can notice easily that the scalar is invariant under shift symmetry: $\phi \to \phi + \text{const}$. So, we can restrict ourselves to the domain of a circle of radius R, so that $\phi \sim \phi + 2\pi R$. The modifications from our previous discussion are:

1) The center of mass momentum π_0 must take values that are integer multiples of $1/R$ in order for $\mathcal{V}_\alpha$ to be well-defined.

2) Boundary condition: when we circle once the cylinder, then φ winds around its field configuration m times, i.e. $\varphi(x+L,t) = \varphi(x,t) + 2\pi mR$.

So, our field reads

$$\varphi_{m,n}(x,t) = \varphi_0 + \frac{n}{4\pi gL}t + \frac{2\pi Rm}{L}x + \frac{i}{\sqrt{4\pi g}}\sum_{k\neq 0}\frac{1}{k}\left(a_k e^{2\pi ik(x-t)/L} - \bar{a}_{-k}e^{2\pi ik(x+t)/L}\right), \tag{12.1}$$

from which, as before, we can get the holomorphic part of the operator $\partial\varphi(z)$

$$i\partial\varphi(z) = \left(\frac{n}{4\pi gR} + \frac{mR}{2}\right)\frac{1}{z} + \frac{1}{\sqrt{4\pi g}}\sum_{k\neq 0} a_k z^{-k-1} \tag{12.2}$$

So, letting $a_0 = \sqrt{4\pi g}\left(\frac{n}{4\pi gR} + \frac{mR}{2}\right)$ and $\bar{a}_0 = \sqrt{4\pi g}\left(\frac{n}{4\pi gR} - \frac{mR}{2}\right)$, we can write

$$i\partial\varphi(z) = \frac{1}{\sqrt{4\pi g}}\sum_{k} a_k z^{-k-1}, \tag{12.3}$$

this is our Laurent expansion for the primary field $\partial\varphi$.

Then, the stress energy tensor is

$$\begin{aligned} T(z) &= -2\pi g : \partial\varphi(z)\partial\varphi(z) : \\ &= \frac{1}{2}\sum_{n,m\in\mathbb{Z}} z^{-n-m-2} : a_n a_m : \\ &= \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \end{aligned}$$

where in the last line we just write the Laurent expansion if T. Equating the last two lines gives us (similar expression for the anti-holomorphic part)

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_{n-m}a_m : \qquad (n\neq 0) \\ L_0 &= \sum_{m>0} a_{-m}a_m + \frac{1}{2}a_0^2. \end{aligned} \tag{12.4}$$

Therefore, recalling (10.43), we have

$$\begin{aligned} H &= \frac{4\pi g}{2gL}a_0^2 + \frac{2\pi}{L}\sum_{n>0}(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n) \\ &= \frac{2\pi}{L}\left[a_0^2 + \sum_{n>0}(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n)\right] \end{aligned} \tag{12.5}$$

and from (12.4)

$$\boxed{H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0\right)} \tag{12.6}$$

Specializing in our case, we have

$$\begin{aligned}
L_0 &= \sum_{k>0} a_{-k}a_k + 2\pi g\left(\frac{n}{4\pi gR} + \frac{mR}{2}\right)^2 \\
\bar{L}_0 &= \sum_{k>0} \bar{a}_{-k}\bar{a}_k + 2\pi g\left(\frac{n}{4\pi gR} - \frac{mR}{2}\right)^2 \\
H &= \frac{2\pi}{L}\sum_{k>0}\left(a_{-k}a_k + \bar{a}_{-k}\bar{a}_k + \frac{n^2}{4\pi gR^2} + m^2R^2\pi g\right).
\end{aligned} \tag{12.7}$$

If we label our vacuum as $|m,n\rangle$, so that $a_k|m,n\rangle = \bar{a}_k|m,n\rangle = 0$ for $k>0$, then

$$\begin{aligned}
L_0|m,n\rangle &= 2\pi g\left(\frac{n}{4\pi gR} + \frac{mR}{2}\right)^2|m,n\rangle = h_{m,n}|m,n\rangle\,, \\
\bar{L}_0|m,n\rangle &= 2\pi g\left(\frac{n}{4\pi gR} - \frac{mR}{2}\right)^2|m,n\rangle = \bar{h}_{m,n}|m,n\rangle\,, \\
(L_0+\bar{L}_0)|m,n\rangle &= \frac{n^2}{4\pi gR^2} + m^2R^2\pi g = \Delta_{m,n}|m,n\rangle\,, \\
(L_0-\bar{L}_0)|m,n\rangle &= nm = s_{m,n}|m,n\rangle\,,
\end{aligned}$$

from where we can read out the conformal dimension, the scaling dimension, and the spin of the vacuum states, respectively.

12.2 The Free Fermion

Recall the action

$$S = \frac{g}{2}\int d^2x\bar{\Psi}\gamma^\mu\partial_\mu\Psi, \qquad \Psi = (\psi,\bar{\psi})^T \tag{12.8}$$

on a cylinder of radius L, we have

$$\psi(x) = \sqrt{\frac{2\pi}{L}}\sum_k b_k e^{2\pi ikx/L},$$

where the operators b_k satisfy the anti-commutation relations $\{b_k, b_q\} = \delta_{k+q,0}$ (recall that this theory represents free Majorana fermions). We can see that $b_0^2 = \frac{1}{2}$. It is important here to distinguish between two boundary conditions:

$$\begin{aligned} \psi(x+L) &= \psi(x), \qquad \text{Ramond } (R), \\ \psi(x+L) &= -\psi(x), \qquad \text{Neveu-Schwarz } (NS). \end{aligned} \tag{12.9}$$

In the (R) sector, k takes integer values, while in the (NS) sector k takes half-integer ones. To carry the time dependence, as before,

$$\begin{aligned} & b_k(t) = b_k e^{-2\pi i k t/L} \\ \Longrightarrow\ & \psi(x,\tau) = \sqrt{\frac{2\pi}{L}} \sum_k b_k e^{-2\pi k(\tau - ix)/L} \end{aligned}$$

and the Hamiltonian can be written as

$$H = \sum_{k>0} \omega_k b_{-k} b_k + E_0, \tag{12.10}$$

where $\omega_k = \frac{2\pi|k|}{L}$. In the (R) sector, there is a zero mode b_0 (recall that $b_0^2 = 1/2$) that leads to a degeneracy of the vacuum; namely, the states $|0\rangle$ and $b_0\,|0\rangle$ have the same energy E_0.

12.2.1 Mapping onto the plane

The results we have obtained so far are in the cylinder. Typically, we work on the plane instead, so it would be nicer to generalize those results for the plane. To do this, we perform the transformation $z \to w^{2\pi w/L}$ and we know

$$\begin{aligned} T_{cyl}(z) &= \left(\frac{2\pi}{L}\, z\right)^2 \left(T_{plane} - \frac{c}{24z^2}\right) \\ \sum_n z^{-n-2} L_n^{cyl} &= \left(\frac{2\pi}{L}\, z\right)^2 \left(\sum_n z^{-n-2} L_n^{plane} - \frac{c}{24z^2}\right), \end{aligned}$$

from where we obtain that

$$L_0^{cyl} = L_0^{plane} - \frac{c}{24}, \tag{12.11}$$

so

$$\boxed{H = \frac{2\pi}{L}\left(L_0^{plane} + \bar{L}_0^{plane} - \frac{c}{12}\right)}. \tag{12.12}$$

In general, L_0^{plane} depends on the boundary conditions. So, coming back to our fermionic example ($c = 1/2$), we use the following results

$$
\begin{aligned}
\langle T(z)\rangle_{plane} &= 0, \qquad && (NS) \\
\langle T(z)\rangle_{plane} &= \frac{1}{16z^2}, \qquad && (R)
\end{aligned} \tag{12.13}
$$

Thus,

$$
\begin{aligned}
L_0^{cyl} &= L_0^{plane} - \frac{1}{48}, \qquad && (NS) \\
L_0^{cyl} &= L_0^{plane} - \frac{1}{48} + \frac{1}{16}, \qquad && (R)
\end{aligned} \tag{12.14}
$$

Let's calculate the quantum Hamiltonian of this model on the plane. As before, we start by computing $T(z)_{plane} = -\pi g : \psi(z)\partial\psi(z) :$, we can first show that the Laurent expansion

$$
\psi(z)_{plane} = \sum_k b_k z^{-k-1/2},
$$

and so the two sectors can be redefined on the plane

$$
\begin{aligned}
\psi(e^{2\pi i}z) &= \sum_{k\in\mathbb{Z}} b_k e^{-2\pi i k} e^{-\pi i} z^{-k-1/2} = -\psi(z), \qquad && (R) \\
\psi(e^{2\pi i}z) &= \sum_{k\in\mathbb{Z}+1/2} b_k e^{-2\pi i k} e^{-\pi i} z^{-k-1/2} = \psi(z), \qquad && (NS)
\end{aligned}
$$

So,

$$
\begin{aligned}
T(z)_{plane} &= \pi g \sum_{k,q} \left(k + \frac{1}{2}\right) z^{-k-3/2} : b_q b_k : z^{-q-1/2} \\
&= \pi g \sum_{n,q} \left(k + \frac{1}{2}\right) z^{-n-2} : b_{n-k} b_k :,
\end{aligned}
$$

which allow us to identify the operators L_n

$$
L_n^{plane} = \pi g \sum_k \left(k + \frac{1}{2}\right) : b_{n-k} b_k :
$$

absorbing the constant we have

$$
\begin{aligned}
L_0^{plane} &= 2\pi g \left(\sum_{k>0} k b_{-k} b_k \right), \qquad && k \in \mathbb{Z} + 1/2 \\
L_0^{plane} &= 2\pi g \left(\sum_{k>0} k b_{-k} b_k + \frac{1}{16} \right), \qquad && k \in \mathbb{Z}
\end{aligned}
$$

So the Hamiltonians for each sector are:

$$\begin{aligned} H^{(NS)} &= \frac{2\pi}{L}\left(L_0^{plane} + \bar{L}_0^{plane} - \frac{1}{24}\right) \\ H^{(R)} &= \frac{2\pi}{L}\left(L_0^{plane} + \bar{L}_0^{plane} + \frac{1}{12}\right). \end{aligned} \tag{12.15}$$

12.3 The 2d Ising model at criticality

The lattice model

$$H = -J\left(\sum_{n=1}^{N-1} X_n X_{n+1} + g\sum_{n=1}^{N} Z_n\right) \tag{12.16}$$

has an exact solution for either Open or Periodic boundary conditions. It's well-known that the point $g = 1$ is a critical point of the system. Following Kogut's paper [19] (Open Boundary Conditions), the Hamiltonian can be written as a fermionic one using Jordan-Wigner transformations. After some algebra, one finds

$$H = \frac{1}{a}\sum_k \Lambda_k \eta_k^\dagger \eta_k + \text{const},$$

where we included the factor of a, the lattice spacing, so that the Hamiltonian has dimensions of energy, we have set $J = 1/2$, and

$$\Lambda_k = \sqrt{1 + g^2 + 2g\cos k},$$

with $\left\{\eta_k^\dagger, \eta_{k'}\right\} = \delta_{k,k'}$. The minimum is located at $k = \pm\pi$, so let's define $k = \pi + aq$, where q has units of momentum. Then when we approach the continuum limit, $a \ll 1$, we have

$$\Lambda_k = \sqrt{1 + g^2 - 2g\cos(aq)} \approx \sqrt{(1-g)^2 + g(aq)^2},$$

before sending a to zero, let's go to the critical point $g = 1$, we obtain

$$H = \sum_q |q|\eta_q^\dagger \eta_q + \text{const},$$

which is the Hamiltonian of a free fermionic model. Furthermore, if we consider our fermions to be Majorana, namely, $\eta_q^\dagger = \eta_{-q}$, then we recover the free (Majorana) fermion Hamiltonian that we review in the previous section.

So, the critical point of the 2d Ising model has the same universality class as the free fermion model, viz. both have central charge $c = 1/2$ and their set of operators have the same set of conformal dimensions.

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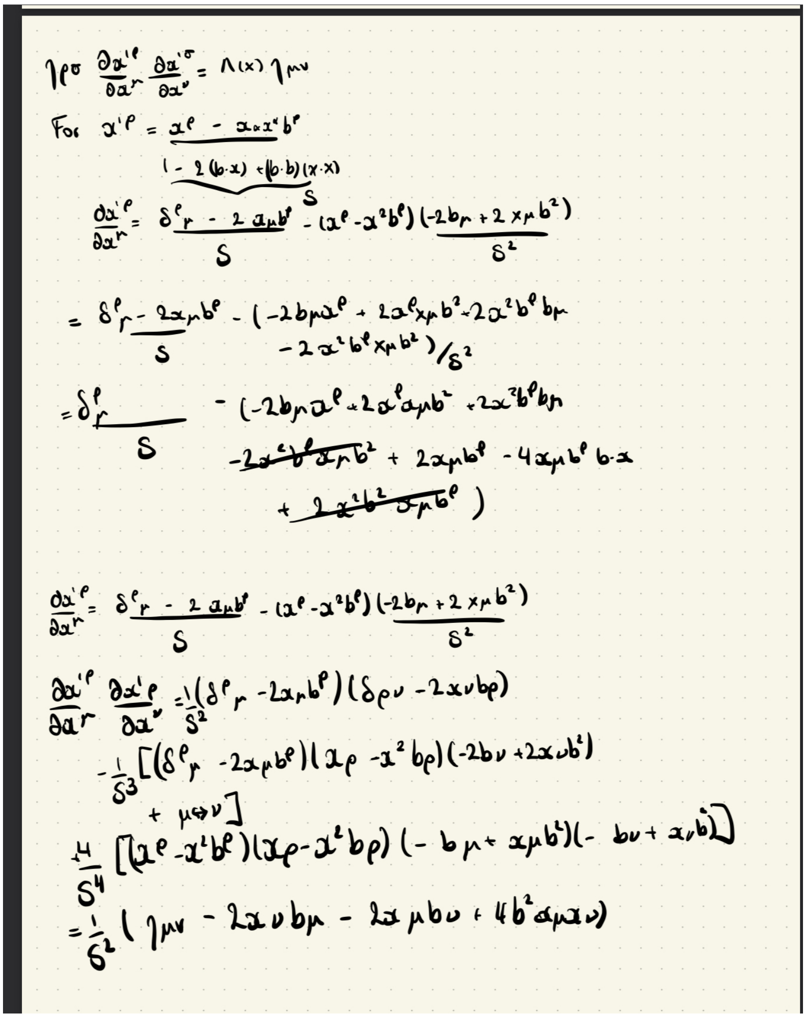
A Derivation of scale factor for special conformal transformations



$$\eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x)\, \eta_{\mu\nu}$$

For $x'^\rho = \dfrac{x^\rho - x_\alpha x^\alpha b^\rho}{\underbrace{1 - 2(b\cdot x) + (b\cdot b)(x\cdot x)}_{S}}$

$$\frac{\partial x'^\rho}{\partial x^\mu} = \frac{\delta^\rho{}_\mu - 2 x_\mu b^\rho}{S} - (x^\rho - x^2 b^\rho)\frac{(-2b_\mu + 2x_\mu b^2)}{S^2}$$

$$= \frac{\delta^\rho{}_\mu - 2x_\mu b^\rho}{S} - (-2b_\mu x^\rho + 2x^\rho x_\mu b^2 + 2x^2 b^\rho b_\mu - 2x^2 b^\rho x_\mu b^2)/S^2$$

$$= \frac{\delta^\rho{}_\mu}{S} - (-2b_\mu x^\rho + 2x^\rho x_\mu b^2 + 2x^2 b^\rho b_\mu \cancel{-2x^2 b^\rho x_\mu b^2} + 2x_\mu b^\rho - 4x_\mu b^\rho\, b\cdot x \cancel{+ 2x^2 b^2 x_\mu b^\rho})$$

$$\frac{\partial x'^\rho}{\partial x^\mu} = \frac{\delta^\rho{}_\mu - 2 x_\mu b^\rho}{S} - (x^\rho - x^2 b^\rho)\frac{(-2b_\mu + 2x_\mu b^2)}{S^2}$$

$$\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'_\rho}{\partial x^\nu} = \frac{1}{S^2}(\delta^\rho{}_\mu - 2x_\mu b^\rho)(\delta_{\rho\nu} - 2x_\nu b_\rho)$$

$$- \frac{1}{S^3}\Big[(\delta^\rho{}_\mu - 2x_\mu b^\rho)(x_\rho - x^2 b_\rho)(-2b_\nu + 2x_\nu b^2) + \mu \leftrightarrow \nu\Big]$$

$$+\frac{4}{S^4}\Big[(x^\rho - x^2 b^\rho)(x_\rho - x^2 b_\rho)(-b_\mu + x_\mu b^2)(-b_\nu + x_\nu b^2)\Big]$$

$$= \frac{1}{S^2}(\eta_{\mu\nu} - 2x_\nu b_\mu - 2x_\mu b_\nu + 4b^2 x_\mu x_\nu)$$

Figure 8. Derivation of scale factor for SCT

$$-\frac{2}{S^3}\Big[\left(x_\mu - x^2 b_\mu - 2x_\mu (b\cdot x) + 2b^2 x^2 x_\mu\right)\left(-b_\nu + x_\nu b^2\right) + \mu\leftrightarrow\nu \Big]$$

$$+\frac{4}{S^4}\Big[\big(\underbrace{x^2 - 2x^2\, b\cdot p - x^4 b^4}_{S x^2}\big)\big(b_\mu b_\nu - b_\mu x_\nu b^2 - x_\mu b_\nu b^2 + x_\mu x_\nu b^4\big)$$

$$= \frac{1}{S^2}\left[\eta_{\mu\nu} - 2x_\nu b_\mu - 2x_\mu b_\nu + 4b^2 x_\mu x_\nu\right]$$

$$-\frac{2}{S^3}\Big[\Big(-x_\mu b_\nu + x_\mu x_\nu b^2 + x^2 b_\mu b_\nu - x^2 b_\mu x_\nu b^2 + 2x_\mu (b\cdot x)\, b_\nu - 2x_\mu x_\nu b^2 (b\cdot x) - 2b^2 x^2 x_\mu b_\nu + 2b^4 x^2 x_\mu x_\nu + \mu\leftrightarrow\nu\Big) - 2x^2\big(b_\mu b_\nu - b_\mu x_\nu b^2 - x_\mu b_\nu b^2 + x_\mu x_\nu b^4\big)\Big]$$

$$= \;{}''\; -\frac{2}{S^3}\Big[x_\mu b_\nu\left(-1 - 3b^2 x^2 + 2(b\cdot x) + 2b^2x^2\right) + x_\nu b_\mu\left(-1 - 3b^2x^2 + 2(b\cdot x) + 2b^2x^2\right) + x_\mu x_\nu b^2\left(2 - 4\, b\cdot x + 4b^2x^2 - 2b^2x^2\right)\Big]$$

$$= \;{}''\; -\frac{2}{S^2}\left[-x_\mu b_\nu - x_\nu b_\mu + 2x_\mu x_\nu b^2\right]$$

$$= \frac{1}{S^2}\Big[\eta_{\mu\nu} \cancel{- 2x_\mu b_\nu - 2x_\nu b_\mu + 4x_\mu x_\nu b^2} \cancel{+ 2x_\mu b_\nu + 2x_\nu b_\mu - 4x_\mu x_\nu b^2}\Big]$$

Figure 9.