2-Flavor Schwinger Model Notes

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This series of notes are mainly compiled from [1]: Coleman '76 (More about the Massive Schwinger Model).

1 Two-Flavor Schwinger Model

Before starting it all, let's recall the bosonization dictionary:

$$\begin{aligned} \text{mass term} \quad & :\bar{\psi}_i\psi_i: \quad \longleftrightarrow \quad -c|m_i|\cos\phi_i \\ \text{vector current} \quad & :\bar{\psi}_i\gamma^\mu\psi_i: \quad \longleftrightarrow \quad \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi_i, \end{aligned} \tag{1}$$

where $c = \frac{e\gamma}{2\pi}$ and there is no sum over $i = 1, 2$. Notice, without loss of generality, that we take the coefficient of the mass term $c|m_i| > 0$; this because we can always shift the field ϕ_i by $\phi_i \pm \pi$ otherwise.

Then, the action of QED_2 with 2 fermions is

$$\begin{aligned} S &= \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{\theta}{2\pi}F_{01} + \sum_{i=1}^{2}\left[i\bar{\psi}_i\gamma^\mu D_\mu\psi_i - m_i\bar{\psi}_i\psi_i\right]\right) \\ &= \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{\theta}{2\pi}F_{01} + \sum_{i=1}^{2}\bar{\psi}_i\left(i\gamma^\mu\partial_\mu - m_i\right)\psi_i + eA_\mu j_V^\mu\right), \end{aligned} \tag{2}$$

with $j_V^\mu = \sum_{i=1}^{2}\bar{\psi}_i\gamma^\mu\psi_i$.

Let, $\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, then we can rewrite the previous action in terms of Ψ:

$$S = \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{\theta}{2\pi}F_{01} + \sum_{i=1}^{2}\Psi^\dagger\left(i\gamma^0\gamma^\mu\partial_\mu - \mathcal{M}\right)\Psi + eA_\mu j_V^\mu\right), \tag{3}$$

and $j_V^\mu = \Psi^\dagger\gamma^0\gamma^\mu\Psi$ and $\mathcal{M} = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}$. From here, we can see that

$$\mathcal{M} = \frac{m_1 + m_2}{2}\,\mathbb{I} + \frac{m_1 - m_2}{2}\,\sigma_3, \tag{4}$$

where σ_3 is the z-Pauli matrix. Therefore, S is invariant under 2 symmetries: an electric $U(1)_E$ gauge symmetry and a global flavor $U(1)_F \supset SU(2)_F$ symmetry. Notice that the full $SU(2)_F$ is broken by the σ_3 term with coefficient proportional to $(m_1 - m_2)$. The action of the symmetries on the fields is

$$\begin{aligned} U(1)_E : \quad & \Psi \to \Psi' = e^{-ie\alpha(x)}\Psi \\ & A_\mu \to A'_\mu = A_\mu - \partial_\mu \alpha(x) \\ U(1)_F : \quad & \Psi \to \Psi' = e^{-i\alpha\sigma_3}\Psi. \end{aligned} \tag{5}$$

In particular, if we take $-m_1 = m_2 = m > 0$, then $\mathcal{M} = -m\sigma_3$ and the theory is invariant under the $U(1)_F$ subgroup of the flavor symmetry. But, if we consider $\mathcal{M} = m\mathbb{I}$, then we have an invariance under the full $SU(2)_F$ symmetry given by the transformation

$$SU(2)_F : \quad \Psi \to \Psi' = e^{-i\vec{\alpha}\cdot\vec{\sigma}}\Psi, \tag{6}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of the 2×2 Pauli matrices.

The corresponding vector currents associated with these symmetries are:

$$\begin{aligned} U(1)_E : \quad & j_E^\mu = \Psi^\dagger \gamma^0 \gamma^\mu \Psi \\ SU(2)_F : \quad & j_F^{a,\mu} = \Psi^\dagger \gamma^0 \gamma^\mu \tau^a \Psi \\ U(1)_F : \quad & j_F^{3,\mu} = \Psi^\dagger \gamma^0 \gamma^\mu \tau^3 \Psi, \end{aligned} \tag{7}$$

with $\tau^a = \frac{\sigma^a}{2}$ and $a = 1, 2, 3$. The first line corresponds to the electric current and the second line describe the isospin currents (the third line is just a special case of the $SU(2)_F$ current).

By making use of the bosonization dictionary, we write the fermionic action as a bosonic one, viz

$$\begin{aligned} S = & \int d^2x \left(\frac{1}{2}F_{01}^2 + e\frac{\theta}{2\pi}F_{01} + \frac{1}{8\pi}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right] + \right. \\ & \left. + \frac{e}{2\pi}A_\mu\epsilon^{\mu\nu}\partial_\nu(\phi_1+\phi_2) + cm_1^2\ sgn(m_1)\cos\phi_1 + cm_2^2\ sgn(m_2)\cos\phi_2 \right) \\ = & \int d^2x \left(\frac{1}{2}F_{01}^2 + e\ \frac{(\phi_1+\phi_2+\theta)}{2\pi}F_{01} + \right. \\ & \left. + \frac{1}{8\pi}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right] + cm_1^2\ sgn(m_1)\cos\phi_1 + cm_2^2\ sgn(m_2)\cos\phi_2 \right) \\ = & \int d^2x \left(\frac{1}{8\pi}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right] - \frac{e^2}{8\pi^2}(\phi_1+\phi_2+\theta)^2 + cm_1^2\ sgn(m_1)\cos\phi_1 + cm_2^2\ sgn(m_2)\cos\phi_2 \right) \\ = & \int d^2x \left[\frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right] - \frac{e^2}{2\pi}\left(\phi_1+\phi_2+\frac{\theta}{2\sqrt{\pi}}\right)^2 + \right. \\ & \left. + cm_1^2\ sgn(m_1)\cos\left(2\sqrt{\pi}\phi_1\right) + cm_2^2\ sgn(m_2)\cos\left(2\sqrt{\pi}\phi_2\right) \right]. \end{aligned} \tag{8}$$

Therefore, the corresponding Hamiltonian is:

$$
\begin{aligned}
H =& \mathcal{N}_m \left[\frac{\Pi_1^2}{2} + \frac{\Pi_2^2}{2} + \frac{(\partial_x \phi_1)^2}{2} + \frac{(\partial_x \phi_2)^2}{2} + \frac{e^2}{2\pi} \left(\phi_1 + \phi_1 + \frac{\theta}{2\sqrt{\pi}} \right)^2 - \right. \\
& \left. - c m_1^2 \, sgn(m_1) \cos\left(2\sqrt{\pi}\phi_1\right) - c m_2^2 \, sgn(m_2) \cos\left(2\sqrt{\pi}\phi_2\right) \right] \\
=& \mathcal{N}_m \left[\frac{\Pi_+^2}{2} + \frac{\Pi_-^2}{2} + \frac{(\partial_x \phi_+)^2}{2} + \frac{(\partial_x \phi_-)^2}{2} + \frac{\mu^2}{2} \phi_+^2 - \right. \\
& \left. - c m_1^2 \, sgn(m_1) \cos\left(\sqrt{2\pi} \left(\phi_+ + \phi_- \right) - \frac{\theta}{2} \right) - c m_2^2 \, sgn(m_2) \cos\left(\sqrt{2\pi} \left(\phi_+ - \phi_- \right) - \frac{\theta}{2} \right) \right],
\end{aligned}
\tag{9}
$$

where $\mu^2 = \frac{2e^2}{\pi}$, $\phi_+ = \frac{1}{\sqrt{2}} \left(\phi_1 + \phi_2 + \frac{\theta}{2\sqrt{\pi}} \right)$, $\phi_- = \frac{1}{\sqrt{2}} \left(\phi_1 - \phi_2 \right)$, and $\mathcal{N}_m(\cdots)$ represents the normal ordering with respect to the masses m_1 and m_2. If we now let

$$
\begin{aligned}
& H^\theta(m_1, m_2) = -c m_1 |m_1| \cos\left(\sqrt{2\pi} \left(\phi_+ + \phi_- \right) - \frac{\theta}{2} \right) - c m_2 |m_2| \cos\left(\sqrt{2\pi} \left(\phi_+ - \phi_- \right) - \frac{\theta}{2} \right) \\
& = -c m_1 |m_1| \left[\cos\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) - \sin\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \sin\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right] - \\
& - c m_2 |m_2| \left[\cos\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) + \sin\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \sin\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right],
\end{aligned}
\tag{10}
$$

here, in the second line we have shifted $\phi_- \to \phi_- + \frac{\theta}{2\sqrt{2\pi}}$.

If we take $m_1 = \eta m$ and $m_2 = m$, with $\eta = \pm 1$ and $m > 0$, then

$$
\begin{aligned}
H^\theta(\eta m, m) = -c m^2 & \left[(1+\eta) \cos\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) + \right. \\
& \left. + (1-\eta) \sin\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \sin\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right].
\end{aligned}
\tag{11}
$$

In particular,

$$
\begin{aligned}
H^{\theta=\pi}(m, m) &= -2 c m^2 \sin \sqrt{2\pi}\phi_+ \sin \sqrt{2\pi}\phi_- \\
H^{\theta=0}(-m, m) &= -2 c m^2 \sin \sqrt{2\pi}\phi_+ \sin \sqrt{2\pi}\phi_-,
\end{aligned}
\tag{12}
$$

that means $H^{\theta=\pi}(m, m) = H^{\theta=0}(-m, m)$.

For simplicity, let's consider $m_1 = m_2 = m$, then the full Hamiltonian reads:

$$
\begin{aligned}
H &= \mathcal{N}_m \left[\frac{\Pi_+^2}{2} + \frac{\Pi_-^2}{2} + \frac{(\partial_x\phi_+)^2}{2} + \frac{(\partial_x\phi_-)^2}{2} + \frac{\mu^2}{2}\phi_+^2 + H^\theta(m,m) \right] \\
&= \mathcal{N}_\mu \left[\frac{\Pi_+^2}{2} + \frac{\Pi_-^2}{2} + \frac{(\partial_x\phi_+)^2}{2} + \frac{(\partial_x\phi_-)^2}{2} + \frac{\mu^2}{2}\phi_+^2 - \right. \\
&\qquad \left. -2cm^2 \left(\frac{\mu^2}{m^2}\right)^{1/4} \cos\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2}\right) \cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2}\right) \right] \\
&= \mathcal{N}_\mu \left[\frac{\Pi_+^2}{2} + \frac{\Pi_-^2}{2} + \frac{(\partial_x\phi_+)^2}{2} + \frac{(\partial_x\phi_-)^2}{2} + \frac{\mu^2}{2}\phi_+^2 - \right. \\
&\qquad \left. -2cm^{3/2}\mu^{1/2} \cos\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2}\right) \cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2}\right) \right],
\end{aligned} \tag{13}
$$

where in the second line, we have used that [2]

$$
\mathcal{N}_m e^{i\beta\phi} = \left(\frac{\mu^2}{m^2}\right)^{\frac{\beta^2}{8\pi}} \mathcal{N}_\mu e^{i\beta\phi}. \tag{14}
$$

Before keep going, let's analyze how do the bosonic fields transform under the symmetries of the fermionic theory:

$$
\begin{aligned}
U(1)_E: &\quad j_E^0 = \Psi^\dagger\Psi = \psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2 = \frac{1}{2\pi}\left(\partial_x\phi_1 + \partial_x\phi_2\right) = \frac{1}{2\pi}\partial_x\phi_+ \\
U(1)_F: &\quad j_F^{3,\mu} = \Psi^\dagger\gamma^0\gamma^\mu\tau^3\Psi = \bar{\psi}_1\gamma^\mu\psi_1 - \bar{\psi}_2\gamma^\mu\psi_2 = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu(\phi_1-\phi_2) = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi_-.
\end{aligned} \tag{15}
$$

We can see two things: 1) the field ϕ_+ transforms trivially under the isospin symmetry, so it is a singlet under of $SU(2)_F$. Then, $j_F^{a,\mu}$ only depends on ϕ_-, an example is shown in the second line, where this current also represents the subgroup $U(1)_F$. 2) From the action (8), we can see that only ϕ_+ interacts with A_μ, this means that ϕ_- is neutral (singlet) under the $U(1)_E$ electric (gauge) symmetry.

If we focus on the case $\mu \gg m$, that is $e \gg m$ (strong coupling limit), then ϕ_+ is very massive and we can project it out, so we study the simpler model

$$
\begin{aligned}
H_{eff} &= \mathcal{N}_m \left[\frac{\Pi_-^2}{2} + \frac{(\partial_x\phi_-)^2}{2} - 2cm^{3/2}\mu^{1/2}\cos\frac{\theta}{2}\cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2}\right) \right] \\
&= \mathcal{N}_{m'} \left[\frac{\Pi_-^2}{2} + \frac{(\partial_x\phi_-)^2}{2} - m'^2\cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2}\right) \right],
\end{aligned} \tag{16}
$$

where we have used (14) with $m' = (2cm\mu^{1/2}\cos\theta/2)^{2/3}$, this to show that we only have one parameter m', or similarly m/e. In fact, $(m'/\mu) = \left[2c(\pi/\sqrt{2})^{1/2}\cos\theta/2\right]^{2/3}(m/e)^{2/3}$. So, the particles we have thrown away are $O(\mu/m') = O(e/m)^{2/3}$ times more massive than the ones we have retained.

The effective Hamiltonian in (16) is of the sine-Gordon type with interactions proportional to $\cos\beta\phi$, and so the corresponding particle spectrum is as follows: the theory contains 2 particles of equal mass, say M, soliton, and antisolitons. Also, from the current (which is trivially conserved)

$$j^{\mu} = \frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_{\nu}\phi,$$

the soliton has charge $+1$ and the antisoliton has charge -1 (the charge here corresponds to the winding number of ϕ). In particular, for $\beta^2 < 4\pi$, the theory has soliton-antisoliton bound states with zero charge and mass given by $M_n = 2M\sin(\beta'^2 n/16)$, with $\beta'^2 = \dfrac{\beta^2}{1-\beta^2/8\pi}$ and $n = 1, 2, \cdots < 8\pi/\beta'^2$.

In our problem, we have $\beta^2 = 2\pi$ and we can identify this current with our $j_F^{3,\mu}$ one for ϕ_-. Therefore, we expect that in the strong coupling limit, we have a soliton and antisoliton solutions with isospin charge ± 1 and mass M. But, additionally, we expect charge-zero isospin solutions which correspond to soliton-antisoliton bound states. In this case, $\beta'^2 = 8\pi/3$ and so, there are 2 bound states with masses[1]

$$M_1 = 2M\sin\pi/6 = M$$
$$M_2 = 2M\sin 2\pi/6 = \sqrt{3}M$$

Hence, the low lying spectrum contains an isotriplet formed by a soliton (isospin charge $+1$), an antisoliton (isospin charge -1), and a soliton-antisoliton bound state (isospin charge 0), all with mass M. Next, we have another soliton-antisoliton bound state (isospin charge 0) with mass $\sqrt{3}M$. If there are other stable particles, they should be $O(e/m)^{2/3}$ times heavier than these (projected out in our strong coupling limit approximation). As a side note, since the only mass parameter is m', then $M \propto m'$.

So far, these results has been stated for the case $\theta \neq \pi$. From (16), it's clear that for the $\theta = \pi$ case, things change. According, to [1], in this case we also obtain a sine-Gordon model, but with a $\beta^2 = 8\pi - O(m/e)^2$. For this model with $4\pi < \beta^2 < 8\pi$, the spectrum only contains solitons and antisolitons (with winding charge $+1$ and -1, respectively). This corresponds to an isodoublet in our Schwinger model counterpart, where the 2 particles have isospin charges $\pm 1/2$.

Therefore, for weak coupling (not studied here, but similar to the 1-flavor Schwinger model case), we have that there are 2 half-asymptotic isodoublets: quark Ψ and antiquark $\Psi^{\dagger}\gamma^0$, with electric charges $+1$ and -1. On the other hand, for strong coupling, there is only one isodoublet electrically neutral (because it corresponds to pure ϕ_- excitations that are singlets under $U(1)_E$). So, it's natural to think that there is a phase transition similar to the one-flavor case for $\theta = \pi$.

2 General version with distinct masses

The Hamiltonian with different masses can be written as a sum of preserving $SU(2)_F$ symmetry and the breaking part, as follows:

[1] Here, this result was obtained relying on the strong coupling approximation, that is, projecting out the massive isosinglet field ϕ_+. However, this approximation is only valid in this limit. A more careful treatment is needed for the generalized Sine-Gordon Hamiltonian (13), which predicts that $M_2/M_1 = 3$ [3].

$$
\begin{aligned}
H =& \mathcal{N}_m \left[\frac{\Pi_1^2}{2} + \frac{\Pi_2^2}{2} + \frac{(\partial_x \phi_1)^2}{2} + \frac{(\partial_x \phi_2)^2}{2} + \frac{e^2}{2\pi} \left(\phi_1 + \phi_1 + \frac{\theta}{2\sqrt{\pi}} \right)^2 - \right. \\
& \qquad - c m_+^2 \; sgn(m_+) \left(\cos\left(2\sqrt{\pi}\phi_1\right) + \cos\left(2\sqrt{\pi}\phi_2\right) \right) - \\
& \qquad\qquad \left. - c m_-^2 \; sgn(m_-) \left(\cos\left(2\sqrt{\pi}\phi_1\right) - \cos\left(2\sqrt{\pi}\phi_2\right) \right) \right] \\
=& \mathcal{N}_m \left[\frac{\Pi_+^2}{2} + \frac{\Pi_-^2}{2} + \frac{(\partial_x \phi_+)^2}{2} + \frac{(\partial_x \phi_-)^2}{2} + \frac{\mu^2}{2} \phi_+^2 - \right. \\
& - 2 c m_+^2 \; sgn(m_+) \left[\cos\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right] + \\
& \qquad \left. + 2 c m_-^2 \; sgn(m_-) \left[\sin\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \sin\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right] \right\} \\
=& \mathcal{N}_\mu \left\{ \frac{\Pi_+^2}{2} + \frac{\Pi_-^2}{2} + \frac{(\partial_x \phi_+)^2}{2} + \frac{(\partial_x \phi_-)^2}{2} + \frac{\mu^2}{2} \phi_+^2 - \right. \\
& - 2 c m_+^2 \left(\frac{\mu^2}{m_+^2} \right)^{1/4} \; sgn(m_+) \left[\cos\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right] + \\
& \qquad \left. + 2 c m_-^2 \left(\frac{\mu^2}{m_-^2} \right)^{1/4} \; sgn(m_-) \left[\sin\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2} \right) \sin\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right] \right\}, \qquad (17)
\end{aligned}
$$

where we have followed the same steps as before (including the shifting of ϕ_-), the equation (14), and we have defined

$$
m_+ = \frac{m_1 + m_2}{2}, \qquad m_- = \frac{m_1 - m_2}{2}.
$$

We can see that we obtain the previous result when we take $m_1 = m_2 = m$, as well as the ones in (12) for different values of θ, m_1, and m_2.

The term proportional to m_- breaks the degeneracy of the isotriplet in the strong coupling limit, and the difference of the masses is proportional to $\frac{m_-^2}{e}$. This is similar to what happens in QCD with the splitting of the masses of $\pi^\pm$ and π^0 due to the difference between the up- and down-quarks [4]. In fact, according to [5], (for $\theta = 0$) the isospin breaking effects are suppressed exponentially in powers of $\exp\left(-(\mu/m_+)^{2/3}\right)$ in the low energy theory (even for $m_- \approx m_+$).

If we restrict ourselves to the $e \gg m_+, m_-$ case, then we can integrate out the heavy boson and get

$$
\begin{aligned}
H_{eff} =& \frac{\Pi_-^2}{2} + \frac{(\partial_x \phi_-)^2}{2} - 2 c m_+^{3/2} \mu^{1/2} \cos\frac{\theta}{2} \; sgn(m_+) \mathcal{N}_{m_+} \left[\cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right] - \\
& \qquad - 2 c m_-^{3/2} \mu^{1/2} \sin\frac{\theta}{2} \; sgn(m_-) \mathcal{N}_{m_-} \left[\sin\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right] \\
=& \mathcal{N}_{m'} \left\{ \frac{\Pi_-^2}{2} + \frac{(\partial_x \phi_-)^2}{2} - m'^2 \; sgn(m_+) \left[\cos\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right] - \right. \\
& \qquad \left. - m'^2 \, \frac{m_-}{m_+} \tan\frac{\theta}{2} \; sgn(m_-) \left[\sin\left(\sqrt{2\pi}\phi_- - \frac{\theta}{2} \right) \right] \right\}, \qquad (18)
\end{aligned}
$$

where $m' = (2 c m_+ \mu^{1/2} \cos\theta/2)^{2/3}$.

References

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